It’s My Maths:
Personalised mathematics learning

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FOREWORD

The theme of the Mathematical Association of Victoria's Annual Conference in 2012 is It's My Maths: Personalised mathematics learning. The effort that mathematics educators make to personalise mathematics for their students is reflected in the conference papers collected here. It seems to be a universal desire of the authors to design mathematical learning opportunities that challenge and inspire their students to take ownership of mathematics and to make it personal.

We are proud to publish an array of contributors whose interests and careers are varied. Their perspectives and range of experience add diversity and interest to these proceedings. This year, as in the past, it is exciting to have a view from beyond Australia as well as a strong national and local focus. Sincere thanks go to the contributing authors whose efforts are reflected in this book and to the reviewers who have offered insightful comments on the papers.

Taking an overview of the collected papers has led me to consider the span of content and the sharing of practice these proceedings represent. A theme of problem solving and investigations can be seen in the conference papers and this theme connects very closely to the issues authors have raised around the use of challenging tasks to stimulate higher order thinking in mathematics. Several writers discuss the central importance of mathematical reasoning. Different types of reasoning are described and examplesare given to help to structure our thinking about the ways in which we can develop students' reasoning in mathematics and in life. For me the link between these papers is the potential of the mathematical task to lead to students being mathematicians.

I have also been interested to read authors' creative ideas describingthe mathematics they use and the ways in which they expand and develop mathematical content with students. A strong theme across the papers is in the use of computer technology to support student learning. It is clear that there are many ways in which technology can enable students to investigate, generalise and conceptualise mathematics.

I hope you find much to challenge and inspire you to personalise mathematics for your students.

Jill Cheeseman

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DIFFERENTIATION: HOW TO REALLY CATER FOR KIDS

Tierney Kennedy

Author

The reality of modern classroom teaching is that no teacher has a class of students who are all working at the one level. Teachers who aim their mathematics lesson at only one or two groups of learners are choosing to believe in a myth that this type of class still exists, if in fact it ever did. Within every single class, even if students are streamed, we have a diversity of learners to cater for. And for a single teacher with 25 or so different students to teach, this can be very difficult to handle. This article shares personal experiences of differentiation and explores practical tips for handling a multi-leveled class.

My Journey Begins:

The difficulty of teaching a multi-leveled group of children first hit home hard for me when I had a grade 5 class in Northern Queensland. I had 31 students in my class. Of these four had significant impairments, and two had reading ages of over 18. Looking back now it was actually a pretty typical class, but I struggled to keep my students all actually learning. I quickly realised that running multiple activities within the one class was not practical unless at least some of the groups were just doing "busy work" and therefore not needing my help. Unfortunately busy work cut down the amount of "learning time" fairly significantly. There had to be a better way!

I decided to approach teaching mathematics in a bit of a back-to-front way and see how we went. Instead of starting with an explanation followed by increasingly difficult questions, I would start with a difficult problem-solving question and then follow up with an explanation only if it was needed. I was trying to use the questions themselves as the learning experience rather than an explanation-based program.

The Problem-Based Process: Our Normal Classroom Practice

I began by asking a few diagnostic questions to find out what my students already knew, often trying to lead them down the garden path a little to work out if they really understood concepts or if they just had the procedures memorised. I tried to find a point where when I asked a question around 80-90% of the kids in my class became a bit stuck. This problem then formed the basis for my lesson, and the rest of the time would be spend with kids trying to solve it without an explanation provided as to how to do it. For support students I kept the problem format and the thinking difficulty the same, simply adjusting the content level down. For my extension students I took the same basic question and added increasingly complex thinking rather than higher level content. This often included working backwards, multiple steps, filling a gap or a non-standard representation. Students worked in pairs or threes on their problems, trying multiple approaches and sharing their thoughts so far with the rest of the class. Together we analysed each idea and compared it to what we already knew of mathematical principles to check that the idea was sound.

This new approach led to some rather surprising discoveries. The first was the sheer volume of mathematics that students could actually work out for themselves when I just encouraged them to give it a go. After six months my traditional explanations virtually vanished from the class as my students learned to generalise their findings and express them as principles, algorithms or formulae. The second surprise was that many fundamental mathematical principles that I believed that the students had worked out years previously turned out to be pretty shaky, and some were missing altogether. This of course had to be dealt with right away. The third, and perhaps most important discovery, was that many of my support students who had previously performed poorly at mathematics often improved so rapidly that most of them were performing at grade standard within 12 months!

As it turned out, many of my support students had been relying on memorisation without understanding underlying concepts. They also had a few fundamental misconceptions about basic principles. Using problem-based teaching allowed these to come out, and challenged students to work out whether or not their ideas were feasible. Once they self-corrected some misconceptions and started generalising about mathematical principles they suddenly started "getting it" and didn't have to rely so heavily on memorisation.

My extension students also started to soar. They were being actually challenged mathematically rather than just being punished with more boring questions. Within six months my class had dramatically improved results at all levels.

Simple Classroom Strategies for Getting Differentiation to Work

A number of very simple strategies can make a big difference to how easy it is to differentiate for your students. Here are my six top ideas for what worked with my students:

1. Support Students

Start with one basic question and then adapt the content in the question down without changing the format. This provides support students with the same thinking challenge, but with content that is more appropriate.

For example:

How many ways can you make the number 372 using hundreds, tens and ones?

- Can you make it using hundreds, tens and ones blocks so that it is still 372?
- How many ways can you make 72 using tens and ones?
- How many ways can you make 23? Is 23 the same as 32? They both have the digits 2 and 3 so are they the same or different? How could you know?
- How many ways can you make 6?

2. Extension students

Start with one basic question and then think of "what if…" scenarios to add complexity for extension students. This increases the difficulty of the thinking skills significantly without necessarily increasing the content load. "What if" questions turn a standard problem into something far more difficult – such as by working backwards, filling a gap, creating more steps or adding complexity.

- What if you could only use 1 hundred to make 372? How many ways could you make it now just using additional tens and ones?
- What if you could only use tenths and hundredths?
- What if you had to start with 1 hundred, 32 tens and 8 ones? What else could you put with it to make 372?
- What if you could only use halves, thirds and quarters to make 372?
- What if you could use three numbers, but two of them had to be the same? How could you make it now?
- What if one of the two numbers had to be 139.7? What are your options for the other numbers?

3. Grouping

I usually end up grouping for behaviour rather than for "ability" or "mixed ability". Usually teachers feel quite anxious about how to choose groups rather than accepting the simple fact that we group according to which students will not try to kill each other. I often allow students to choose their own groups (maximum of 3 in a group), as long as no one is excluded and they are all working. If any group misbehaves then they are split up. It is quite a lot of fun to be able to use the threatened loss of mathematics privileges as a behaviour management tool!

4. Creating Flexible Space Using a Challenge Table

Keep one spare table in your room with about four chairs at it. This allows you to have a very flexible space to cater for different students as they need it. I use this set up in a number of different scenarios:

1. Once I have set a problem: "If you understand what to do, you can go back and work with your partners now. If not, come to the challenge table and we'll have a talk about it". This stops random wandering by students who have no idea how to start.
2. Once we have come back together and shared ideas, and students have realised that their initial ideas were not ever going to work and they need to try something different: "If you want to change your mind you can go back and work with your partners now. If you think you are right, come to the challenge table." This then leaves me with two distinct groups: the ones who are totally wrong and have no idea, and the ones who are right or pretty close to being right. I usually send the group who are right/almost right to the challenge table to work out who is right, or to complete a "what if..." question while I work with those who still have misconceptions.
3. When I cross-reference names with observations it often appears that I seem to consistently miss particular students. These are usually the quiet students who seem to blend into the background rather than grabbing for attention. When I see this happening, I invite those students to work with me at the challenge table for five minutes so that I can check on where they are up to.
4. I like to call students with similar communicating styles to work at the challenge table, such as all those who are quiet and just agree with everyone else in their group. When a group is formed with just those students, one of them eventually has to try something to solve the problem. Alternatively, try calling all the dominant students to fight it out amongst themselves. That frees up the other students to think for themselves.

5. Helping Everyone at the Same Time

One of the difficulties of posing a problem just beyond what students know is that everyone needs help at the same time! To help deal with this overload I use Tip Cards, numbered and blue-tacked to the board, for a problem. The tips should increase in the amount of help that they offer to groups. A group of students who become stuck while working on a problem can decide to go and get one of the tips to give them a clue. They record the tip number on their books. They can then try to solve the problem again, or decide to get another tip. Usually they want to get as few tips as possible, so they work together pretty well to try and solve the problem without getting more cards.

For example (Kennedy, 2011, p.47):

- Is there a different combination of blocks that you could use so that you would still have 372? Can you make it without using 3 hundreds, 7 tens and 2 ones and still make it to be the same size?
- How many hundreds do you have? If you only had 2 hundreds blocks, is there a way that you could use other blocks as well so that you could still make 372?
- Let's just look at the 72. How many tens blocks are there? How many ones blocks are there? Is there another combination of blocks that you could use to make 72?
- Make 6 tens and 12 ones on one side and 7 tens and 2 ones on another side. Are these amounts the same? How do you know? If you line them all up into a straight line will they be the same length? What does that tell you?

6. Teach Students to Self-Differentiate in Lessons

Once I have determined the base problem (where 80-90% of students are a bit stuck), I use Differentiated Problems written in different colours on the whiteboard and allow students to work out which problem they want to try and solve. The base level problem goes in the middle, two above and two below are also present. Each student has to solve at least two levels of the problems. This way everyone can start on the base level problem, drop down if it is too hard, and go up if it is too easy. Once they solve one problem they automatically need to go up a level and try the harder question as well.

Once I started using problem-based teaching to differentiate it quickly becomes addictive. It is just so exciting to see students at all levels being challenged to think mathematically and actually work something out for themselves. Differentiation is a lot of fun when done in combination with a problem-based approach, but in a traditional classroom it can be both difficult and time-consuming.

Reference

Kennedy, T (2011). *Back-to-front maths*, Grade 3 teaching resource book. Townsville, Queensland: Kennedy Press.

RATIO: NEW IDEAS – AN OLD TOPIC

Robert Money

Education Consultant

This paper discusses learning strategies appropriate for two types of ratio problems. The use of ratio tables is explored as a strategy for solving the fixed-ratio problems that occur widely across the curriculum. For problems involving changing ratios the paper discusses how students might benefit from being able to categorize different problem sub-types and choose appropriate solution strategies, in particular the use of geometric modelling using ratio blocks.

Ratio Tables – for Common Ratio Problems

The 2002 Yearbook of the National Council of Teachers of Mathematics (Littwiller & Bright, 2002) was a key document amongst many devoted to *Making sense of fractions, ratio and proportions.* The use of ratio tables has been much less discussed previously (Middleton & van den Heuvel-Panhuizen. 1995; Van der Van der Valk, Wijers, & Frederik, 2000; Money, 2011) and their potential for use in numeracy across the curriculum has not been fully developed.

For example, as part of an 'Estimation walks' activity (HREF2: Maths 300) a numeracy focussed teacher – perhaps even a Phys. Ed. Teacher - might ask "70 paces in 40 seconds to walk the 50 metre track would be how far in one minute?" In response, students with different levels of proficiency in proportional reasoning might develop the following ratio tables:-

Table 1. Ratio Table Solutions to 'Estimation Walks' Activity

Student 1	a	$b = \frac{a}{2}$	$c = 3 \times b$
metres	50	25	75
seconds	40	20	60
paces	70	35	105

Student 2	a	$b = \frac{a}{2}$	$c = a + b$
metres	50	25	$50 + 25 = 75$
seconds	40	20	$40 + 20 = 60$
paces	70	35	$70 + 35 = 105$

Student 3	a	$b = a \times \frac{3}{2}$
metres	50	75
seconds	40	60
paces	70	105

Student 4	a	$b = a \div 40$	$c = b \times 60$
metres	50	1.25	75
seconds	40	1	60
paces	70	1.75	105

Class discussion could lead to the top row entries, which explain the different thinking processes of the four students. Thankfully, no student has made the mistake of adding a constant, say 20, to each column. All students use the first property of ratio tables - that you can multiply or divide as you fill in new columns.

- Students 1 and 3 show a good understanding of the proportional relationships involved.
- Student 2 could well be limited to multiplicative halving and doubling strategies. This fits well with focussing attention on finding distances and paces for the remaining 20 seconds of the minute. The student uses the second important property of ratio tables: 'You can add multiples'. The procedure of the last column is justified by the distributive law - in this case $n \times 40 + n \times 20 = n \times (40 + 20)$. The student may not realize the use here of the more sophisticated property: If $\frac{p}{q} = \frac{r}{s}$, then $\frac{p+r}{q+s}$ is an equivalent fraction.
- Student 4 has used the 'unitary method' strategy, possibly with the aid of a calculator. The method could well have been taught previously, with the chances of success improved by the decimals involved being relatively simple.

Ratio tables have the following nine advantages over other approaches to constant ratio problems:

1. They do not involve the algebra of proportion equations, but scaffold for it.
2. They encourage exploration, with different methods reaching the same correct solution.
3. They provide a record of the student's thinking, valuable for the teacher as diagnostic and assessment information.
4. Their use can be explained in terms of the important properties of equivalent fractions and the distributive law.
5. They provide a simple means of representing a problem and focussing attention on the target: for example 'We need to get a 100 in the metres row.'
6. They can be used to generalize approaches, in particular to assist students in dealing with more challenging arithmetic. (See below.)
7. The row and column headings help students keep track of the quantities and the operations involved in what they are doing.
8. They can be extended to extra rows, involving ratios of more than two quantities.
9. Their use can complement learning in a spreadsheet environment in which the 'Copy down' facility supports the multiplicative requirements of ratio tables.

Extending the Table

A number of further questions could arise from discussion of the results of the above 'Estimation walks' activity.

Time in hours? Distance in kilometres from home to the centre of town?

In consequence, students could structure their approach to these problems by extending their ratio tables by providing extra rows for 'minutes', 'hours' and 'kilometres' and by entering the initial conversion factors within the table.

Table 2. Extended ratio table for the 'Estimation walks' activity

metres	50	75					1000	
seconds	40	60						
paces	70	105						
minutes		1				60		
hours						1		
kilometres							1	5

Changing Ratio Problems

A standard approach to problems involving change of ratio is to use algebra, presumably with learning algebra as the goal that motivates a teacher-directed approach. Curriculum in Singapore in particular (Koay & Fwe, 2003; Ministry of Education, Ministry of Education, 2006; Musa & Malone, 2011) also emphasizes approaches involving 'ratio blocks' and systematic trials – in particular for problems requiring whole number solutions.

These problems can be fitted into five categories according to what is the given information and what is required.

- Type 1: Same change for both quantities

Example: The ratio of Josie's age to her father's age is 2:7. 15 years later she will be half his age. How old is Josie now?

Harder example: Mum's age to grandmother's age is 3:5. 4 years later the ratio of their ages is 5: 8. How old is mum now?

- Type 2: One quantity remains the same

Example: A drinks mixer starts with 200 ml of a drink containing 3 parts fruit juice to 2 parts water. How much extra water would be needed to make it 1 part fruit juice to 3 parts water?

Harder example: A drinks mixer starts with 200 ml of a drink containing 3 parts fruit juice to 2 parts water. How much extra water would be needed to make it 2 parts fruit juice to 3 parts water?

- Type 3: Total quantity unchanged

Type 3a: Final ratio given

Example: Ben and Jack shared some footy cards in the ratio 1:3. When Jack gave 8 of his cards to Ben, the new ratio of Ben's cards to Jack's was 2:5. How many cards did Jack have at first?

Type 3b: Final ratio not given

Example: Ben and Jack shared some footy cards in the ratio 1:3. After Ben gave one quarter of his footy cards to Jack, Jack had 80 more cards than Ben. How many footy cards did they have altogether?

- Type 4: Given changes to quantities

Example: Julie and Jo owned savings in the ratio 4: 3. Over the next week Julie increased her savings by $10 and Jo increased hers by $12, bringing the ratio of their savings to 5: 4. How much money did each girl end up with?

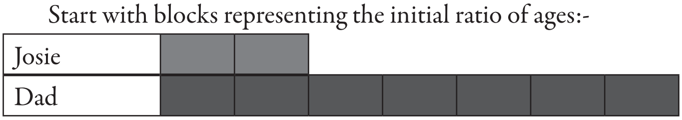
- Type 5: Ratios of ratios

Natural number example (with more than one answer.)
The ratio of the number of boys to the number of girls in a class is 3:2. If each boy and each girl is given stickers in the ratio 3:4, a total of 510 stickers are needed. How many boys and how many girls could there be in the class?
Continuous number example (with an infinite number of answers.)
The ratio of mango juice to lemon juice in a fruit drink mix is 3:2. Straight mango juice requires just three quarters of the amount of sugar as does straight lemon juice. If 510g of sugar is to be used, how much mango juice and how much lemon juice are needed in the fruit drink mix?

Research (Musa & Malone, 2011) has provided support for the argument that student understanding and performance on such ratio problems is improved when their attention is focused on recognition of such categories. This accords with general theories on Case Based Reasoning and Cognitively Guided Instruction and is not surprising, since the non-algebraic approaches emphasized in the Singapore curriculum are appropriate to Types 1 through 3 as listed above.

Consider the Type 1 'age difference' problem above. Since a whole number solution is required perhaps the best solution strategy is 'systematic guess and check'. However, geometric modelling using 'ratio blocks' (HREF3: Thinking blocks) can be successfully applied:-

Start with blocks representing the initial ratio of ages:-



Then add equal blocks to each row, until Josie's blocks are half dad's.

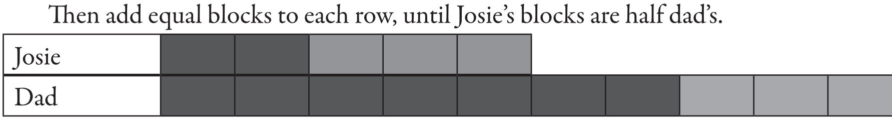


Figure 1. Block model for a Type 1 problem.

The model (Figure 1) is completed once the student recognizes that Josie's 5 blocks are half of dad's 10.

The three extra blocks required represent the extra 15 years, so each block represents 5 years. From this it follows that Josie's age now is 2 x 5 = 10.

For Type 2 problems blocks are added to one row until the required final ratio is obtained. For the 'drinks mixer' example above, start with the initial 3: 2 ratio represented as 3 blocks juice and 2 blocks water, with each block representing 40 ml (Figure 2).

Juice	40	40	40
Water	40	40	

Then add blocks to the water until the 9 water blocks are recognized to be 3 times the number of juice blocks.

Juice	40	40	40
Water	40	40	

Figure 2. Block model for a Type 2 problem.

The extra water represented by the 7 blocks is therefore 7 x 40 = 280 ml.

For Type 3 problems blocks need to be moved from one row to the other. For the first of the 'Ben and Jack sharing' problems above, students need to recognize that the number of blocks to be shared must be a common multiple of 4 (1 + 3) and of 7 (2 + 5), namely 28.

Start with a total of 28 blocks shared in the ratio 1: 3, with 7 for Ben and 21 for Jack (Figure 3).-

Once one block is moved from Jack's row to Ben's the student needs to recognize Ben's 8 blocks as being two fifths of Jack's 20 blocks.

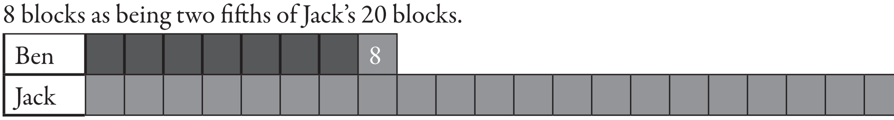


Figure 3. Block model for a Type 3 problem.

The one block moved represents 8 footy cards, so Jack must have started with 21 x 8 = 168 cards.

Problems of the above three types are presented at the 'Thinking Blocks' website (HREF 3). The following screenshot (Figure 4) shows the half way stage of the solution of a Type 2 problem. The block model has been built and now the student needs to interpret the values involved.

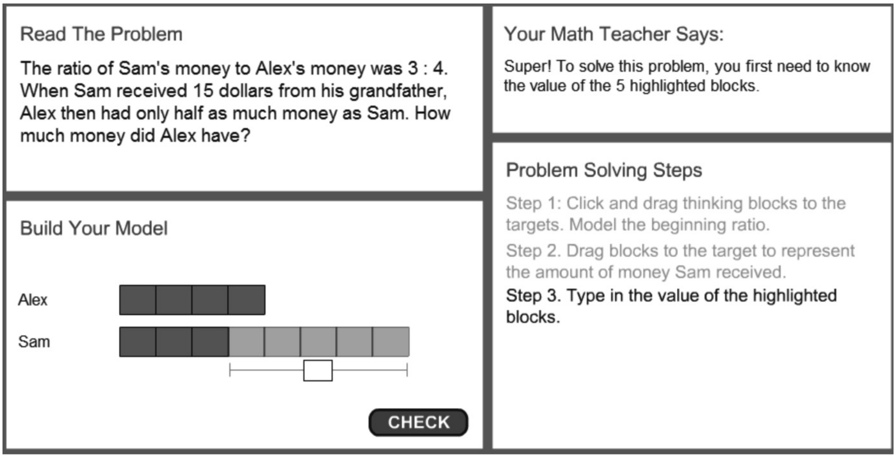


Figure 4. Screenshot from http://www.thinkingblocks.com/ThinkingBlocks_Ratios/TB_Ratio_Main.html

Ratio in the Curriculum

Ratio tables are becoming increasingly used in mathematics classes. The next task for mathematics teachers is to introduce colleagues in other faculties to the powerful way in which ratio tables can support many of the numeracy opportunities that arise in their classrooms. Ratio tables can provide an excellent starting point for inter-disciplinary dialogue and stronger cross-curricular links.

The use of ratio blocks in tackling quite complex ratio problems is a feature of the primary curriculum in Singapore. This example can be followed in our Australian middle school student-centred classrooms, with case based reasoning used to categorize and solve the different types of problem. Algebraic approaches to such problems can be addressed at an appropriate time, but experience with ratio block models and case based reasoning can be powerful supports of conceptual understanding and procedural skill in this context.

Current Australian curriculum documents (HREF1: ACARA) contain only brief descriptions of ratio. Curriculum for Singapore, by contrast, not only specifies the inclusion of changing ratio problems but specifies the various different sub-types of such problems. The draft Australian Mathematics curriculum mentions constant ratio in Year 6 and again in Year 8. Proportion is not introduced until Year 10. Rates, ratio and proportion make up one of four topics in Unit 1 (Year 11) of the draft Australian Senior Mathematics Course B: General Mathematics. In the context of these limited references there is a need for a strong

Australian research effort on the teaching of ratio, ratio tables in particular. Encouraging results from other countries (e.g. The Netherlands, Singapore, and USA) need to be tested in the Australian classroom environment. Classroom teachers can play their part in calling for this research effort and in having it undertaken in their classrooms.

References

Beswick, K. (2011). Making your own paint chart. In *Australian Mathematics Teacher* 67(1), 6-11.

Gomez, G. (2002). Multiplicative reasoning: Developing student's shared meanings. In B. Littwiller & G. Bright (Eds.), *Making sense of fractions, ratio and proportions. 2002 Yearbook.* of the National Council of Teachers of Mathematics. Reston, VA: NCTM.

Kent, B., et al. (2002). Using representational contexts to support multiplicative reasoning. In B. Littwiller & G. Bright (Eds.). *Making sense of fractions, ratio and proportions. 2002 Yearbook* of the National Council of Teachers of Mathematics. Reston, VA: NCTM.

Koay, P. L., & Fwe, Y. S. (2003). The teaching and learning of ratio and proportion in Singapore primary schools. In B. Clarke, A. Bishop, R. Cameron, H. Forgasz & W. T. Seah (Eds.), *Making mathematicians, Proceedings of the 40th Conference of the Mathematical Association of Victoria*. Melbourne: MAV.

Middleton, J. A., & van den Heuvel-Panhuizen. (1995). The ratio table. *Mathematics in the Middle School, 1*(4), 282 – 287.

Ministry of Education (2006). *Primary mathematics syllabus.* Singapore: Curriculum Planning and Development Division.

Musa, N., & Malone, J. (2011). The use of problem categorization in the learning of ratio. In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S.Thornton (Eds.), *Mathematics: Traditions and [new] practices. Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia and the Australian Association of Mathematics Teachers.* Adelaide: AAMT and MERGA.

Money, R. (2011). Ratio: tables and length models. In L. Bragg (Ed.), *Maths is multi-dimensional. Proceedings of the 48th Conference of the Mathematical Association of Victoria*. Melbourne: MAV.

Shield, M., & Dole, S. (2008). Proportion in middle school mathematics: It's everywhere. *Australian Mathematics Teacher.* 64(3) 10-15.

Van der Valk, T., Wijers, M., & Frederik, I. (2000). *Using ratio tables from mathematics in secondary school science. Proceedings of the 25th annual ATEE conference, Barcelona. Retrieved from www.fi.uu.nl/bps/.../ATEE2000science%20ratio%20tables.html*

Weinberg, S. (2002). Proportional reasoning; one problem, many solutions. In B. Littwiller & G. Bright (Eds.), *Making sense of fractions, ratio and proportions 2002 Yearbook,* the National Council of Teachers of Mathematics: Reston, VA: NCTM.

HREF1: ACARA: Australian Mathematics Curriculum Foundation to Year 10 version 2.0. Retrieved from

http://www.australiancurriculum.edu.au/Mathematics /Curriculum/F-10

HREF2: Maths 300: Estimating walks. Retrieved from

http://www.maths300.esa.edu.au/index.php?option=com_content&view=article&id=115

HREF3: Thinking blocks: Ratio. Retrieved from

http://www.thinkingblocks.com/ThinkingBlocks_Ratios/TB_Ratio_Main.html

MATHEMATICA™: PANDORA'S BOX OR CLASSROOM EMPOWERMENT?

Brenton R Groves

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The amount of Mathematica™ material on the web is huge: the alphabetical site index, MathWorld - 13,086 entries, over 8000 interactive demonstrations under the Computable Document Format (CDF), tutorials and howtos, forums, blogs and The Virtual Book (10,000 pages). Many teachers not connected with Wolfram have contributed essays and books on the web. Everything can be downloaded for free from the Internet and Mathematica™ is not required.

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The presentation consists of three modules:

1. Wolfram Research Web Resources (Main 'Visual' Page);
    a. Wolfram Blog
    b. Wolfram Demonstration Project
    c. MathWorld
    d. Forums
    e. Alphabetical Site Index
2. Wolfram Computable Document Format (CDF) Player:
    a. Download free CDF Player.
    b. Three Examples – Demonstration Project
3. Resources outside Wolfram Research:

Wolfram Web Resources (Main Menu):

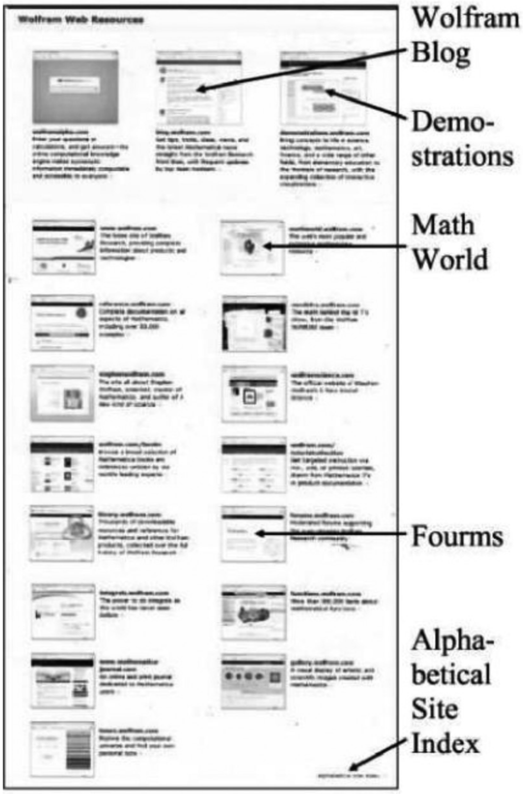

Figure 1 Wolfram Resources Main Menu

Ctrl/Click on http://www.wolfram.com/webresources.html There are 18 icons plus the link to the alphaindex at the lower right hand of the page.

Four of these icons and the alphaindex are of direct interest to teachers.

blog.wolfram.com Get tips, tricks, ideas, views, and the latest *Mathematica* news straight from the Wolfram Research front lines, with frequent updates by top team members.

http://demonstrations.wolfram.com/ Bring concepts to life in science, technology, mathematics, art, finance, and a wide range of other fields, from elementary education to the frontiers of research, with this expanding collection of interactive visualizations. See CDF Player below.

Wolfram Math World (13,086 entries) http://mathworld.wolfram.com/ Search for any math topic, *Hotel Room* for example. This is the old problem of a missing dollar for three men renting a hotel room. Direct URL: http://mathworld.wolfram.com/MissingDollarParadox.html, http://www.wolfram.com/support/community/ or forums.wolfram.com

Moderated forums supporting the ever-growing Wolfram Research community.

> *MathGroup* Forum is a moderated email list and internet newsgroup. Members number in the many thousands from all over the world and include most of the leading experts in the use of Mathematica *MathGroup* archives dating back to 1989 are also available.
>
> Student Support Forum Post your queries about using *Mathematica* to this forum or browse previous posts for answers to questions that others have posted.
>
> Faculty Program Forum Connect and share ideas with other educators through the Wolfram Faculty Program. Any teacher intending to use *Mathematica*™ resources in their teaching should join this forum. http://www.wolfram.com/faculty/ The Wolfram Faculty Program Membership Portal has the details on joining.

The alphabetical site index' at the extreme lower right-hand side of the main menu is equivalent to a search-able library card catalog of all Wolfram documentation. Direct URL: http://reference.wolfram.com/alphaindex/A.html

CDF Player Download:

Wolfram Research's Computable Document Format (CDF) is going to become a universal textbook and journal article format. It allows an author to insert interactive graphics and visuals and create hyperlinks into a PDF style format that runs under any operating system. Complete details can be found at http://www.wolfram.com/cdf/ .

This web page lets you download a free Computable Document Format (CDF) reader which can display any animated CDF file or a *Mathematica* notebook with a static version of its source code. Download the program to your hard disk from http://www.wolfram.com/cdf-player/ and then bookmark the program or create a CDF shortcut icon in your *Mathematica* folder.

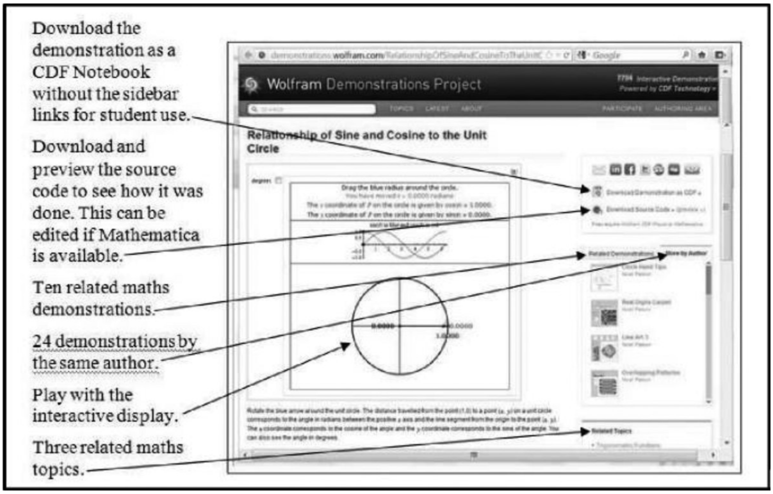

Figure 2. One of 7787 Interactive Demonstrations Powered by CDF.

Demonstrations:

1. Click on CDF icon or bookmark to bring up the Wolfram CDF Player. Then click on 'Welcome to Wolfram CDF Player' under the 'Help' palette.
2. The Welcome page lists two slides on the mid right-hand side. Click on 2/2 and then the 'Get Started' box to bring up the Wolfram Demonstration Project listing some 7332 interactive demonstrations. The index is at the bottom of the screen.
3. Or direct URL: http://demonstrations.wolfram.com/
4. You can download the demonstration as a CDF file which will run under the CDF Player or as a *Mathematica* notebook .nb with the source code. The CDF Player will open .nb files but they are not animated or modifiable.

Example 1:

http://demonstrations.wolfram.com/RelationshipOfSineAndCosineToTheUnitCircle/

Below the image are three 'snapshots' that can be cut and pasted into any word document if a computer is not available.

Example 2: 'Euclid's Pythagorean Theorem Proof'

Search for 'Proof Pythagorean Theorem' in the search box at the top of the screen for an example. Eleven demonstration CDF files are given. Click on 'Euclid's Proof of the Pythagorean Theorem' (#5) to bring up the CDF file and run the interactive demo. Direct URL: http://demonstrations.wolfram.com/EuclidsProofOfThePythagoreanTheorem/

Example 3: 'Wrapping the number line around the unit circle

Return to the demonstration program web page http://demonstrations.wolfram.com/ and search for "Unit Circle Number Line'. 18 examples returned. Click on 'Wrapping the number line around the unit circle' (#1). Direct URL:
http://demonstrations.wolfram.com/WrappingTheNumberLineAroundTheUnitCircle/

This demonstration should convince any student that 2 π is a better choice than 360 degrees when calculating trigonometric (or circular if you wish) functions.

Resources outside Wolfram Research:

Googling '*Mathematica* secondary education – Wolfram' brings up hundreds of applications by teachers and academics who have their own web pages. Typical entries:

HREF1: Oliver Knill, Harvard University. Three neat *Mathematica* demos for teaching. With source code. Uses video animation. http://www.math.harvard.edu/~knill/pedagogy/techdemo/mathematica/

HREF2: Sadri Hassani, *Mathematical Methods Using Mathematica: For Students of Physics and Related Fields*, 239 pages. http://202.38.64.11/~jmy/documents/ebooks/Hassani_Mathematical_Methods_Using_Mathematica,Springer.pdf Use browser if hyperlink fails.

HREF3: Andrew Schultz, Math 103: Matrix Theory with Applications, Complete subject documentation and test sheets. Stanford University. http://palmer.wellesley.edu/~aschultz/summer06/math103/

HREF4: James J Kelly, Essential *Mathematica* for Students of Science: Tutorial

Approach to Mastery of *Mathematica*, University of Maryland. A massive effort of value to anyone writing course material using Mathematica. http://www.physics.umd.edu/courses/CourseWare/EssentialMathematica/

Additional References:

HREF5: *Mathematica Learning Path for Primary and Secondary Educators Learn Mathematica or advance your expertise with high-level resources, from tutorials to videos and online training, specifically curated for primary and secondary educators.* http://www.wolfram.com/support/learn/primary-secondary¬education.html

HREF6: *Free Online Seminar Catalog Learn about Mathematica, the Computable Document Format (CDF), and other Wolfram technologies.* Wolfram Training courses include quick starts to cover the basics and in-depth looks at concepts and applications. Watch courses on demand according to your schedule or join live courses online, in a classroom, or at your organization. http://www.wolfram.com/services/education/seminars/

HREF7: *All 1084 secondary school educational demonstrations, displayed 29 to a page.* http://demonstrations.wolfram.com/education.html?edutag=High+School+Mathematics&limit=20

The following are details of the first major CDF mathematics text, the wave of the future:

HREF8: *Courseware of the Future—Today: Developing Interactive Textbooks with CDF*. A 2:52minute video by Eric Schulz, Co-author of Calculus by Briggs and Cochran.

> He says, "It's way beyond a regular word processor. It's beyond web pages and applets and Java..." Using the free Wolfram CDF Player, students can immediately navigate through sections and explore the ebook's interesting interactive figures and intuitive text, which combine to bring hard-to-convey concepts to life. "Those that have been teaching have been yearning for something that would bring our subject alive and move beyond the textual content we normally find, and support that with visualizations that are interactive,"
>
> http://www.wolfram.com/cdf/information-kit/developing-interactive-textbooks-with-cdf.html

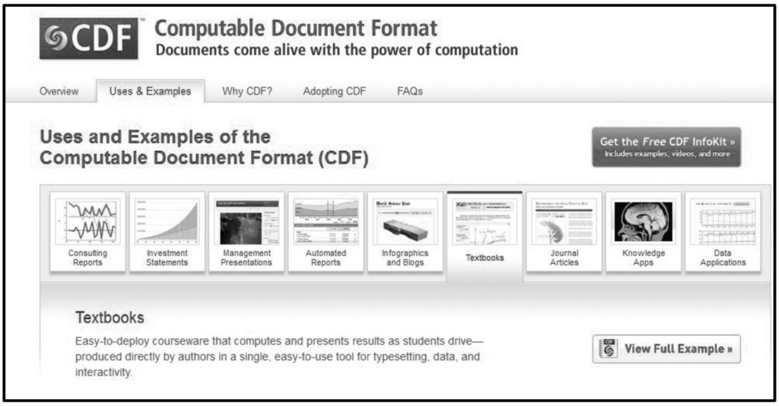

Figure 3. Wolfram Web Page: Uses and Examples of the Computable Document Format (CDF)

HREF9: Open http://www.wolfram.com/cdf/uses-examples/journal-articles.html

Each of the square boxes gives an abstract of a CDF example. The one of most interest to teachers (shown above) would be 'textbooks'. Clicking on 'View Full Example' downloads the complete CDF file

HREF10: Offer of classroom materials for mathematics teachers from the publisher.

http://www.pearsonhighered.com/briggscochran1einfo/detail/learn/index.html

Comments on this Presentation:

A PowerPoint slide show of this paper with images and teaching notes, which was presented at the MAV 2012 Annual Conference, can be downloaded from:

http://dl.dropbox.com/u/49383304/MAV.2012.Teaching.ppsx

Comments and criticisms on this paper are welcome and will be acknowledged.

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Disclaimer:

I have no financial gain from the sale and use of the Mathematica program in any context. The education division of Wolfram Research in the US supplied me with a complimentary copy of *Mathematica* 8 for the preparation of this presentation. The opinions expressed herein are my own and cannot be blamed on anyone else. Brenton R Groves.

RANTS (AND RAVES!): RICH ALGEBRA AND NUMBER TASKS

Lorraine Day

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Rich tasks, incorporating open-ended questions and investigations, can be used to expose students to alternative representations, reasoning and approaches to problem solving leading to deeper understanding. Students who are encouraged to look for patterns in their answers will discover rules and make meaning of them rather than trying to memorise rules that have no meaning for them. The Number and Algebra Strand of the Australian Mathematics Curriculum provides an opportunity for the development of rich tasks to link algebraic reasoning and arithmetical thinking to develop them simultaneously. The process of personalising, contextualising and adapting existing tasks to ensure they are rich, relevant and have the Mathematical Proficiencies embedded provides further opportunities. There are some great tasks and puzzles available that can be used as the catalyst for developing tasks that reflect your personality and interests and those of your students.

Introduction

In the Australian Curriculum: Mathematics, Number and Algebra are developed together, as they complement and enrich each other (ACARA, 2010). Engaging with a variety of contextualised problems in a structured manner, using varied representations to encourage students to reason algebraically, find generalisations and justify their solutions, will enable complementary development of algebraic and arithmetical thinking (Siemon et al., 2011).

Generalisation is at the heart of mathematics and consequently providing students with many opportunities to generalise should be central to mathematics teaching and learning (ACARA, 2010; Siemon et al., 2011). Continual emphasis on finding computational answers to arithmetic problems, rather than an emphasis on explaining and justifying the operations used in finding solutions, can stifle the development of algebraic reasoning (Warren, 2002). It is easier for students to generalise when they actively engage with a problem (Warren & Cooper, 2008) within a meaningful context.

It is far more efficient for students to work with patterns, hypothesise and to test conjectures rather than trying to memorise rules for working with numbers. Students who are encouraged to look for patterns and structure in their answers will discover the rules and make meaning of them rather than trying to remember the rules that have no meaning (Siemon et al., 2011). It is important for students to build on their understanding of the number system to describe relationships and form generalisations (ACARA, 2010) and this cannot be done if there is no understanding on which to build.

Rich tasks, open-ended questions and investigations can be used to expose students to alternative representations and approaches to problem solving, reasoning and understanding. Concentration on the 'big ideas' of Number and Algebra will allow teachers to ensure that their focus is on the important aspects of a topic rather than incidental skills (Small, 2009).

Mathematically Rich Tasks

Mathematically rich tasks have the opportunity to transform school mathematics from a collection of memorised rules and procedures into a vibrant, connected subject where there is an opportunity to develop an understanding and exploration of mathematical concepts (Piggott, 2010). They allow the students to 'get inside' the mathematics (Hewson, n.d.). They do this by having the ability to reach most learners where their known understandings meet the unknown (Ferguson, 2009) or the Zone of Proximal Development (Vygotsky, 1978).

There have been numerous lists published of the characteristics of rich tasks (For example Piggott, 2010; Hewson, n.d.). The best rich tasks allow students to work mathematically, and see others working mathematically, by allowing them to get started and explore, while still providing opportunities for challenge and extension. By their openness these rich tasks have multiple entry and multiple exit points that cater for student diversity. Within meaningful and intriguing contexts they develop thinking, reasoning and communicating skills while seeking genuine understandings. Rich mathematical tasks cater for a variety of learning styles, while encouraging multi-dimensional learning. They highlight the interdisciplinary connections both within and outside of mathematics and use ICT support effectively (Day & Lovitt, 2010). They encourage students to explain their thinking which in turn reveals the depth of understanding (Siemon et al., 2011), which make rich tasks ideal tools to support assessment.

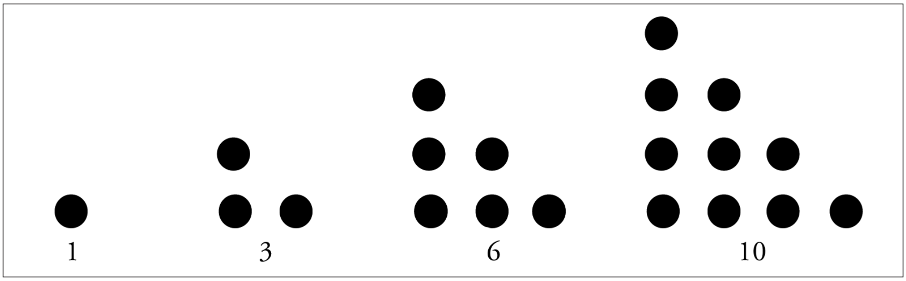
By itself no task is inherently rich. It is the environment in which the teacher presents them, the support and open-ended questioning that is utilised and the expectations placed on the students that makes them rich (Piggott, 2010). By employing a large repertoire of teaching and questioning strategies and expecting students to become co-constructors of their own meaning, teachers can add to the richness of tasks (Day & Lovitt, 2010). Furthermore, teachers who encourage students to challenge themselves and reflect on their learning add to the richness of the learning outcomes (Piggott, 2010).

Designing Rich Tasks

There are a plethora of puzzles, problems and textbook questions which have the potential to be recast as rich tasks. When redesigning tasks it is useful to have a framework in mind, such as a working mathematically process with a concrete-representational-abstract (Allsopp, Kyger & Lovin, 2007) pedagogical structure embedded.

A typical textbook question such as shown in Figure1 provides a starting point for redesigning a problem into a rich task. This task as stated is closed. To convert the question to a rich task requires it to be more investigative to encourage thinking, reasoning and communicating. This can be achieved by establishing a need to create and test hypotheses through using a range of problem solving strategies and extending the task to include a generalisation that can be explained and justified by the students. An opportunity to encourage the use of concrete materials prior to the diagrammatic representation and then moving to the abstract is present and would enhance the richness of the task while making it more accessible for a variety of students.

The first four triangular numbers are shown by this pattern of dots.



Copy the pattern into your book and add the next two triangular numbers to your drawing.

What is the fifth triangular number?

What is the sixth triangular number?

Copy and complete the table below.

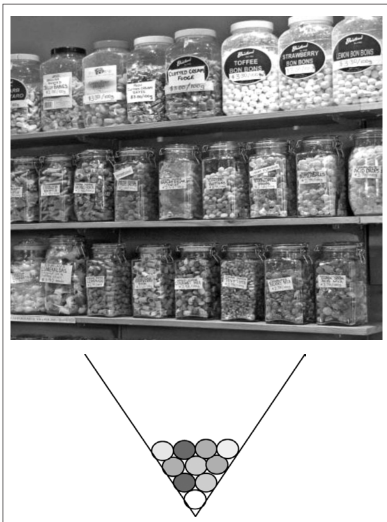
See if you can find a pattern to save time.

Triangular number	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}	8^{th}
Number of dots	1	3	6					

Figure 1. Textbook question 1 (Kammann, 1998, p. 168).

A redesigned task may look something like Figure 2. This task is set in the context of a lolly shop which is meaningful for most students. It sets the scene for students to explore the problem using concrete materials. The task is easily extended to provide opportunities for students to examine equivalent expressions, substitute values into expressions, work backwards to solve linear equations and to look for a generalisation.

The closed task which could be answered by drawing and counting or additive thinking can be transformed into a task where algebraic reasoning and many of the big ideas of number and algebra can work together to establish a rich mathematical experience. The task is easy to begin for all students and can be extended as far as a diagrammatic justification for the general formula for triangular numbers.



Mrs Brown runs an old fashioned lolly shop where she stores the lollies she makes in large jars.

When customers come into the shop and ask for a certain number of lollies, Mrs Brown uses a triangular frame to help her to quickly work out how many lollies she has.

The frame is so old that the numbers on the side of the frame have worn off. Your job is to help Mrs Brown work out what the numbers on the frame should be.

Figure 2. The Old Fashioned Lolly Shop.

The initial statement (see Figure 2) of the Old Fashioned Lolly Shop task is just the starting point of the rich task, and is just the tip of the iceberg. The statement enables students to get started. In order to elicit deeper thinking about this task, it is important that open-ended questions which invite students to demonstrate their understanding are asked. Examples of questions of this type are:

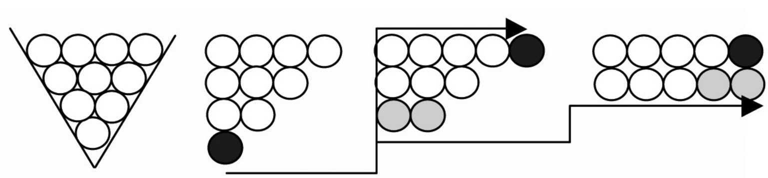
> "How would we work out the total number of lollies in a frame which holds a hundred rows of lollies?"

> "A customer bought 470 lollies for a birthday party and he remembered that Mrs Brown filled her frame and then added 5 more lollies. Can you explain how you would work out how big the frame was that Mrs Brown used?"

> "If I tell you how many rows in a frame, can you tell me how you would work out how many lollies it would hold? Is there a rule for how we work it out? Can you explain your rule in words? Can you show me why your rule works by making a model or drawing a picture? Are you able to write your rule in symbols?"

Another powerful and important concept in the Number and Algebra strand is that of equivalence. Quite often, in an investigative setting, different groups of students see generalisations in differing ways (see Figure 3). This provides an opportunity for students to determine that all of the correct generalisations are equivalent even though they may look different.

Concrete Manipualtion



$n = 4$

$$T_4 = \frac{4}{2} \times (4 + 1)$$
$$= 2 \times 5$$
$$= 10$$

The triangular number you are looking for is the number of rows multiplied by half the number of the next row's number. Or ...

The triangular number you are looking for is the row number multiplied by the next row's number divided by two.

The nth triangular number = $\frac{n}{2} \times (n+1)$

$$T_{10} = \frac{10}{2} \times 11$$
$$= 55$$

The nth triangular number = $n \times \frac{n+1}{2}$

$$T_{10} = 10 \times \frac{11}{2}$$
$$= 55$$

The nth triangular number = $\frac{n(n+1)}{2}$

$$T_{10} = \frac{10 \times 11}{2}$$
$$= 55$$

Figure 3. Different ways of seeing the triangular numbers' generalisation.

The possibilities of redesigning tasks to include the elements of rich investigative tasks are endless. With experience, it becomes easier to look at a routine question and see how it may be developed, especially if it can be linked to a meaningful or authentic context. As we know that students should be exposed to similar types of problems within a variety of contexts (Polya, 1957), it is sometimes a matter of changing the context of a problem that allows it to be opened up. A further example of a textbook question is illustrated in Figure 4.

The theatre restaurant problem

Streeton Heights Senior High School is organising a theatre restaurant night to raise money for the new school gymnasium. The organisers need to work out how many people they can fit into the hall. The tables can be arranged in the formation shown.

How many people could sit at the tables if there are seventy-five tables which need to be arranged in three rows of equal length?

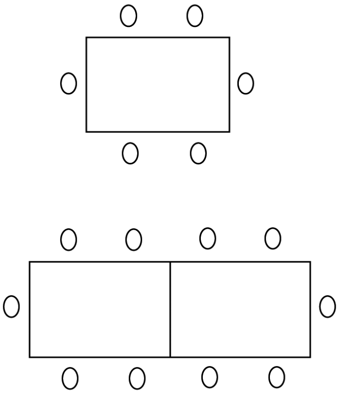


Figure 4. Textbook problem 2 (Kammann, 1998, p. 150).

By altering the context and opening up the task, it is possible to include several aspects of a rich investigative task (see Figure 5). This particular example is situated within an authentic context in East Perth, although it is easily transferrable to other parts of the country.

The Long Table Dinner

In East Perth each year the restaurants at Claisebrook Cove have a Long Table Dinner along the side of the cove. Imagine you have been put in charge of the organisation of next year's event. How are you going to work out how many tables you will need and the number of guests you can seat?

Figure 5. The Long Table Dinner.

Once again, this stimulus is only a starting point for the task. The richness is explored by guiding students and asking appropriate open questions. By carefully orchestrating the questions, it is possible to cover many of the big ideas in algebra within this one task so that its richness is exposed. Among these big ideas are the concepts of a variable, generalisation and function. Substitution and solution of equations can be used to solve problems about the number of guests who can be seated and the number of tables needed for certain numbers of guests. Equivalence can be examined from the way in which the students see the generalisation. The linear function can be graphed and graphical solutions to equations can be explored. Even the concepts of domain and range, as well as discrete and continuous functions, are able to be introduced in an informal manner. Further, simply rearranging the tables into various other configurations, would provide many available opportunities for extending the mathematics.

Conclusion

The linking of number and algebraic thinking in the Australian Curriculum: Mathematics provides an opportunity for mathematical teaching and learning to become more concerned with patterns, relationships and generalisations rather than facts, skills and rules without meaning (Mulligan, Cavanaugh & Keanan-Brown, 2012). The inclusion of the Proficiency strands of understanding, fluency, problem solving and reasoning highlights that rich tasks which link algebraic reasoning to arithmetical thinking through the development of thinking, reasoning, communication and justification are an appropriate mode for teaching and learning mathematics. Modifying tasks to model working like a mathematician within meaningful contexts is one way in which the link between number and algebra can be made more meaningful for our students.

References

ACARA. (2010). *Australian curriculum: Mathematics.* Retrieved from http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10

Allsopp, D., Kyger, M., & Lovin, L. (2007). *Teaching mathematics meaningfully: Solutions for reaching struggling learners.* Baltimore: Brookes Publishing.

Day, L., & Lovitt, C. (2010). *Effectively teaching mathematics.* Unpublished article.

Ferguson, S. (2009). *Same task, different paths: Catering for student diversity in the mathematics classroom.* Australian Primary Mathematics Classroom, 14(2), 32-36.

Hewson, S. (n.d.). *What is a mathematically rich task?* Retrieved from http://nrich.maths.org/6299

Kammann, I. (1998). Heinemann outcomes: *Mathematics for Western Australia* 1. Melbourne, Vic: Heinemann.

Mulligan, J., Cavanaugh, M., & Keanan-Brown, D. (2012). *The role of algebra and early algebraic reasoning in the Australian Curriculum: Mathematics.* In B. Atweh, M. Goos, R. Jorgensen & D. Siemon (Eds.), Engaging the Australian National Curriculum: Mathematics - Perspectives from the field: Online Publication: Mathematics Education Research Group of Australasia. Retrieved from http://www.merga.net.au/node/223

Piggott, J. (2010). *Rich tasks and contexts.* Retrieved from http://nrich.maths.org/5662

Polya, G. (1957). *How to solve it (2nd ed.)*. Garden City, NJ: Doubleday Anchor Books.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R., & Warren, E. (2011). *Teaching mathematics: Foundations to middle years.* Melbourne, Vic: Oxford University Press.

Small, M. (2009). *Great ways to differentiate mathematics instruction.* New York; Reston, VA: Teachers College Press; NCTM.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Warren, E. (2002). *Children's understanding of turnarounds: A foundation for algebra.* Australian Primary Mathematics Classroom, 7(1), 9-15.

Warren, E., & Cooper, T. J. (2008). *Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds thinking.* Education Studies in Mathematics, 67(2), 171-185.

REASONING FOR PROOFS, PATTERNS AND IDEAS

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Reasoning is one of the proficiency strands in the Australian curriculum. This paper looks at the four main types of reasoning (deduction, induction, abduction and analogy) and explains what they are and their value. It uses classroom problems as examples to show where the different types of reasoning occur. These four types of reasoning are for the production of ideas, the production of patterns, and the production of proofs.

Introduction

Reasoning has always been important in mathematics, so the Australian curriculum emphasises reasoning by including it as one of the proficiency strands. The curriculum (HREF1) says this about reasoning.

> "Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. Students are reasoning mathematically when they explain their thinking, when they deduce and justify strategies used and conclusions reached, when they adapt the known to the unknown, when they transfer learning from one context to another, when they prove that something is true or false and when they compare and contrast related ideas and explain their choices."

In this paper we will define and give classroom examples of four basic types of reasoning. The names are only important for an adult audience so we will delay their introduction, but the concepts they stand for are fundamental to the way that mathematics develops. The types of reasoning assist in the production of ideas, the production of patterns, and the production of justifications. Reasoning in these ways is used in all parts of human life. In mathematics they are more explicit. Other examples and explanations are given by Holton, Stacey and Fitzsimons (2012).

Knowing about these forms of reasoning will help teachers understand where a student is in their learning, in their solving of a problem, and in their ability to prove mathematical results. Particular words that students use may act as cues to help the observer see what reasoning is being or has been employed. We will list these as we talk about each type of reasoning. Knowing about these types will help teachers move students forward in their overall knowledge of mathematics.

Reasoning for Proofs

We begin with reasoning for proofs because this is an important feature of mathematics, which distinguishes it from other subjects. In mathematics, given the truth of the axioms and the rules of logic, statements can be proved beyond any doubt. Proof in mathematics is more secure than proof in any other area of human activity. We also start with reasoning for proofs as it shows a basic structure that we will use in explaining the other types of reasoning.

The most famous example of deductive reasoning, dating back to Ancient Greece, is the following. "Socrates is a man; all men are mortal; so Socrates is mortal". Deductive reasoning (see Reid & Knipping, 2010) consists of a 'case' which is a basic statement. This is followed by a 'rule' that can be applied to the case. As a consequence of the rule acting on the statement we get another statement called the 'result'. The case is Socrates is a man, the rule is all men are mortal and the result is Socrates is mortal. Deductive reasoning is at the heart of mathematics. It is the reasoning that proves every step of a mathematical proof. Suppose we want to prove that a triangle (see Figure 1a) with vertices at the centre of a circle O and on its circumference (points A and B) is always an isosceles triangle (i.e. has two sides the same length). The bones of this reasoning are shown in Figure 2.

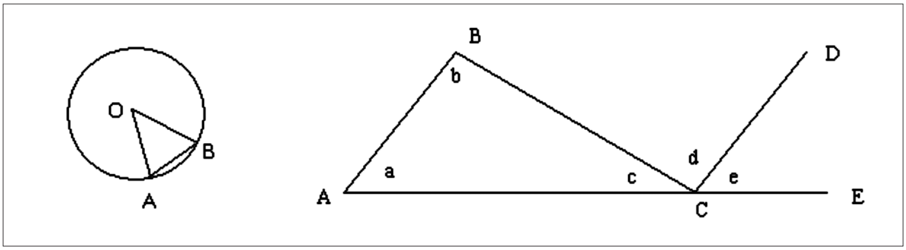

Figure 1. A triangle in a circle (1a) and diagram to find angle sum of triangle (1b)

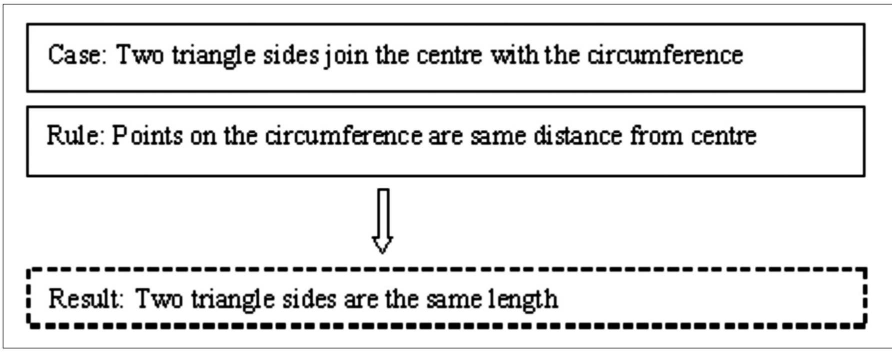

Figure 2. The bones of deductive reasoning

In most mathematical proofs, deduction is used more than once and more than one rule is used. For example, deduction is applied four times to show that the angles in a triangle add to 180°. Consider Figure 1b. Suppose a triangle has angles a, b and c. First we set up the proof by extending the side AC to E and drawing CD parallel to A.

First application of deduction: Angles a and e are corresponding angles (case). Corresponding angles on parallel lines are equal (rule). So a = e (result).

Second application of deduction: Angles b and d are alternate angles (case). Alternate angles on parallel lines are equal (rule). So b = d (result).

Third application of deduction: Angles c, d and e are angles on one side of a line (case). The angles on one side of a line sum to 180° (rule). So c + d + e = 180° (result).

Fourth application of deduction: a = e and b = d (case). c + d + e = 180° (rule). So, replacing equals by equals, a + b + c = 180° (result).

Words that indicate that deductive reasoning has occurred are 'if ... then ...', 'therefore' and 'so'. On the other hand, if you ask your students to 'solve' something it is possible that they may need to engage in deductive behavior. But it depends on what they have to solve. For instance, if this is the first time that they have to solve a linear equation (2x + 1 = 7, say), then they will have to carefully deduce that 2x = 6 and then that x = 3 justified by doing the same operations to equal quantities. However, if they have done this many times the algorithm for solving linear equations will be embedded in their brain and they will solve it automatically. So with 'solve' the task is more important than the word.

Students can gain practice in deductive reasoning from an early age if teachers simply ask them to justify their statements. Even carrying out mental arithmetic such as working out 23 + 45 = 68 or 134 – 99 = 35 are good candidates for justification. At first there is no need for them to write down their arguments. All that is necessary is for them to verbally explain what it is they are doing and why they can do it. Justifying statements is important at any point in schooling but as they grow older, students should be asked to write down their justifications more formally. Simultaneously they should be asked to write proofs for ever more difficult pieces of mathematics.

Deductive reasoning seems to be naturally present in children (and even animals). They use it at an early age in their normal life. Enabling them to be more fluent in this type of reasoning provides them with a skill that can be used in areas outside mathematics. Mathematical deduction is particularly precise and explicit.

Reasoning for patterns

When trying to find patterns, inductive reasoning, also called induction, is often used. (Readers who have studied advanced mathematics should note that this is not 'mathematical induction'.) This type of reasoning still uses case, rule and result (and usually more than one case and result). For deduction, the result is to be found. For induction, the rule is to be found. So induction uses the same three elements in a different order to deduction This is because we are trying to find rules (another name for pattern). We show this in Figure 3, which is in the context of trying to find a rule for the difference between consecutive square numbers.

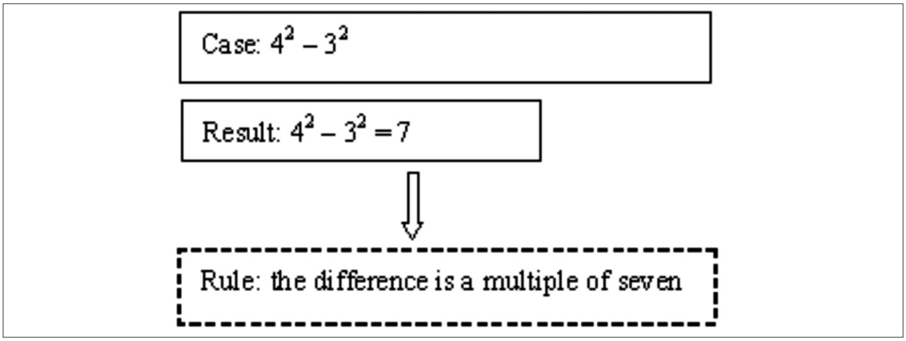

Figure 3. The bones of inductive reasoning

Guessing a rule from only one case as in Figure 3 is obviously not wise, so usually many cases and results are examined to gather evidence for the rule. Induction is not always going to produce correct rules. The rule shown in Figure 3, for example, is easily shown to be false by considering other cases and results, such as

$$2^2 - 1^2 = 3$$
$$3^2 - 2^2 = 5$$

$4^2 - 3^2 = 7$
$5^2 - 4^2 = 9$

The cases here are the differences between consecutive squares. The results are the numbers after the equals signs. Many rules could be inducted to fit this data and make predictions, for example, differences between consecutive squares are always odd e.g. '$66^2 - 65^2$ is going to be an odd number.' But if you look a bit closer, you might induct the rule that the difference between consecutive squares is the sum of the two numbers, predicting correctly that $66^2 - 65^2 = 131$. This is where induction stops. If we want to prove that our possible rule is true, we need to argue deductively. In this case, the result can be proved using the algebraic identity that $a^2 - b^2 = (a-b)(a+b)$. Because the squares are of consecutive numbers, $a - b = 1$ and so $a^2 - b^2 = (a-b)(a+b) = 1.(a+b) = a+b$.

Induction is good for conjecturing (discovering) rules, but there is no guarantee that the rules that have been conjectured are in fact true. It is necessary to go back to deduction to settle this issue. But before results can be proved in mathematics they have to be discovered, so induction is essential. The more uses of the pair 'case' and 'result' you use, the better the chances of getting a correct rule.

'Investigate', 'explore', and 'find' are words that suggest induction needs to be used. They imply that several cases may need to be looked at and a rule found to describe the situation.

Children can be encouraged to use and develop inductive reasoning by looking for patterns in mathematics. This can start by looking for patterns in number. For example, what is special about multiples of 5? (They end in 0 or 5.) It can then develop through such things as what is the next number in the sequence 4, 7, 10, 13, ...? In later years students could conjecture what the m and c in y = mx + c stand for by examining cases (e.g. y = 2x - 6) and results (the graphs). Results proposed by induction should always be followed by answering the question 'why?' As we demonstrated in Figure 3, many rules can be supported by induction, but they are not all true.

Once again, inductive thinking is not confined to mathematics. People are always looking for patterns in life. For example, 'how does a person react to this kind of situation?' And often this is followed by the question 'why?' that enables them to understand why the particular reaction occurs.

Reasoning for Ideas

The final types of reasoning that we consider here are the types that produce ideas. The first of these is described in the diagram in Figure 4, where there is another permutation of case, rule, result. This is abduction. This time it is the case that is wanted. This may seem a strange thing to want to find out but think about any detective story you have ever heard. You have a murder, the result, and all the commonsense and scientific rules of how things work that you need. What you need is a 'case', in this case, the murderer.

In Figure 4, the 'result' (the evidence of the size 9 boots) plus some commonsense and scientific 'rules' leads to a 'case'. This abductive logic is far from watertight and there would need to be a lot more evidence to convict Jack. What we know for sure is the reverse deduction: that "if Jack is the murderer, and the murderer made the footprints, then the footprints are from size 9 boots".

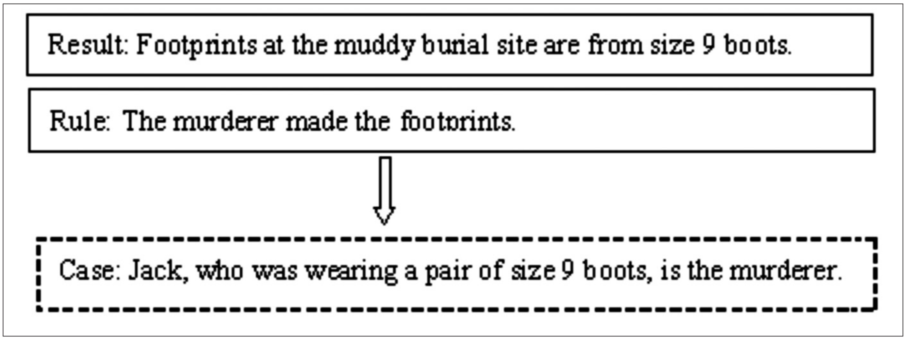

Figure 4. The bones of abductive reasoning

In mathematics what you often need is an idea. Something that might give you a direction to work forward from to solve the problem you are working on. You have some result plus all the rules of mathematics to use but you need an explanation, or something that will set you on the road to a deductive proof. Consider the example of the graph in Figure 5 that shows the weight of a mouse over 11 weeks. How can we explain the shape of the graph? Now the shape of this graph is a result and we have any of the rules of logic and mathematics to use to find the case – what happened to make this graph. There are lots of possibilities so we can't be sure of the case that we abduct, but some possibilities are:

- The mouse had a baby at time T.
- The mouse had an operation to remove a large tumour at time T.
- The mouse was made to run on a treadmill at time T.

To help determine which of these may actually happened, a few questions are in order. Is the mouse female? If not we can move immediately onto the second and third possibilities. Does the 'weeks' axis shown pass through the zero on the weight axis? If not, then the loss of weight might be relatively small compared to the weight of the mouse. In which instance, the third possibility might be likely. Like induction, abduction does not provide definite proof.

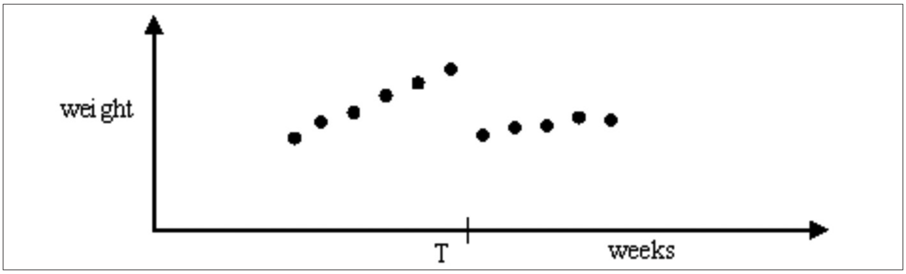

Figure 5. The weight of a mouse over 11 weeks

Note that abduction differs from induction in that induction produces a rule while abduction produces a case, a statement only, nothing that is general. Abductive reasoning is about generating ideas none of which can be guaranteed to be true. They will, however, give a basis for possible deductive follow up work. The detective would later start with 'if the murderer is X' to see if all of the evidence is then consistent. A mathematician, a lot of whose work is like that of a detective, will use this new idea produced by abduction to see if it enables a result to be proved by deduction.

A problem with abduction for teachers is knowing exactly when it has happened. In a problem solving situation our brains generate and reject a large number of possibilities. It is very hard to keep track of these. Unless we are in discussion with someone or they write down their thoughts, we are unlikely to even recognize that abduction has occurred. Words such as 'perhaps', 'maybe' or 'possibly' may show abduction has occurred. But they may equally well be used in inductive or analogical situations.

One way to encourage the development of abductive reasoning, is to give students results and ask them where they came from. The classical case is 'Fred got the number 8 from doing some arithmetic. How did he do that?' As students go higher up the year levels 'arithmetic' could be changed to 'algebra' and later even 'differentiation'. But we have also seen that giving students a graph and asking what produced it is also a good method for developing abductive reasoning, where the relevant rules are both mathematical (how the graph represents the quantities involved) and commonsense. Once again, this kind of thinking is not only valuable in school mathematics. If a factory is suddenly producing unwanted results, then abductive thinking can be used to suggest what the problem might be and therefore begin to fix it.

The fourth basic type of reasoning, and the second in this section, is analogical reasoning (English & Halford, 1995). This is not described using the elements of case, rule and result. Basically we have a situation that we need to resolve. If we know another situation that we can resolve and which is similar to our new situation, then the way of solving the old situation might be used to solve the new one. We show this in Figure 6. In the diagram the new situation has similarities (represented by the squiggles) with the old one. We know how to solve the old situation. Perhaps the same or similar methods will solve the new situation.

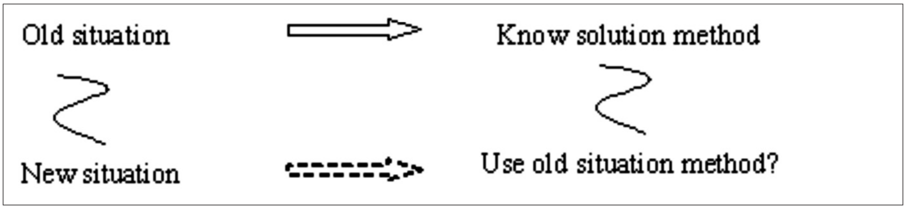

Figure 6. The basis of analogical reasoning

Textbooks thrive on analogical reasoning. Typically they solve a particular problem, for instance, solve $2x + 1 = 7$. Then they provide several problems of the $ax + b = c$ type for the student to solve. The student uses the book's method of solution to solve each of these particular problems, using analogical reasoning.

Using models to illustrate mathematical principles is another common use of analogical reasoning in school mathematics. If we divide a paper circle into quarters, and then halve the quarters, we can see that each resulting piece is an eighth of the whole circle. This is the 'old situation'. In the 'new situation' of calculation with abstract numbers, which the students are to learn, we reason by analogy that

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

One way to solve a problem is to look around for problems that are similar to the new problem in some way. Then the methods of the old problem might be tried to see if they can solve the new problem. So analogical reasoning can be used to find possible solution methods. Mathematicians use analogical reasoning to produce extensions and generalisations of problems. There they build onto an old problem and hope that the same method of solution will work in this new situation.

Conclusion

This paper has looked at the four basic types of reasoning and given school level examples of them. They can be developed over a student's school life by encouraging their use and by noting them explicitly as students progress through increasingly more advanced work. Being fundamental to both mathematics and life as a whole, these reasoning skills are important and should be given attention in classrooms. The easiest way to do this is to include problem solving situations in normal classroom teaching and to emphasise why mathematical results are true. The special place of deduction and proof in mathematics is supremely important, but other forms of reasoning help to generate the patterns and ideas for solving problems.

References

English, L. D., & Halford, G. S. (1995). *Mathematical education: Models and processes*. New Jersey: Lawrence Erlbaum Associates.

Reid, D. A., & Knipping, C. (2010). *Proof in mathematics education: Research, learning and teaching.* Rotterdam: Sense.

Holton, D., Stacey, K., & FitzSimons, G. (2012). *Reasoning: A dog's tale.* Australian Mathematics Teacher, 68(3), 22 - 25.

HREF1: *The Australian Curriculum (Mathematics). (2012).* Retrieved from http://www.australiancurriculum.edu.au/Mathematics/Rationale

AN INTRODUCTION TO MARKOV CHAINS

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Markov chains are mathematical models that use concepts from probability to describe how a system changes from one state to another. The basic ideas were developed by the Russian mathematician A. A. Markov about 100 years ago. These days, Markov chains arise in Year 12 mathematics. This paper offers a brief introduction to Markov chains. A notable feature is a selection of applications that show how these models are useful in applied mathematics.

Introduction

Markov chains are useful mathematical models that use concepts from probability and matrix algebra. In recent years, they have appeared as a topic in Year 12 mathematics subjects in Victoria. Yet many teachers may not have encountered these modelsduring their university education. The school text books set out the ideas required in the elementary methods used in Markov chains; see Coffey et al. (2009) as an example. We will not repeat these ideas here; rather we hope that this paper complements the material found in text books.

We have two aims in writing this article. First, the article provides a general introduction to the basic ideas in Markov chains that may appeal to teachers who have had little experience in studying, or teaching, this topic. Second, the article contains a selection of applications of Markov chains across several fields of knowledge. We hope that these applications illustrate how Markov chains can be useful in contemporary applied mathematics. The basic ideas concerning Markov chains will become clearer through considering specific applications.

Definition of Markov Chain

For the purposes of this paper, we define a Markov chain as follows. More advanced texts will have more general definitions.

Definition: *A Markov chain (MC) is a sequence of random variables* $X = \{X_0, X_1, X_2, \ldots\}$ *with the following properties. For each* $\in \{0,1,2,\ldots\}$, X_k *, is defined on the sample space and* Ω *takes values in a finite set* S. Thus, S $X_k : \Omega \to S$. Also, for $k \in \{0,1,2,\ldots\}$, and $\{i, j, i_{k-1}, i_{k-2}, \ldots, i_0\} \subseteq S$

$P\{X_{k+1} = j \mid X_k = i, X_{k-1} = i_{k-1}, X_{k-2} = i_{k-2}, \ldots, X_0 = i_0\} = P\{X_{k+1} = j \mid X_k = i\}$ (1)

and the transition probabilities

$P\{X_{k+1} = j \mid X_k = i\} = p_{i,j}$ *are independent of* k. *(2)*

This definition is unavoidably technical. We will now tease out the definition in less technical terms.

Essentially we have a system that changes from one state to another over time. The various states of the system are contained in the set S. The set S is called the set of states or state space of the Markov chain X. If the set S has m elements, then X is called an m - state Markov chain.

In Year 12 mathematics, one would tend to consider only a 2-state Markov chain; that is, one in which $m = 2$. Although this simplifies the mathematics, applications of 2-state Markov chains appear to be stilted and contrived. More realistic applications involve more states as we will see below.

Usually k is the discrete variable that measures time. Thus X_k is the state of the system at time k. In particular, X_0 denotes the initial state of the system.

Condition (1) means that, if the system is in state i at time k, then the probability that it will change to state j at time $k+1$ does not depend on what happened in earlier times $k-1, k-2, \ldots, 0$. This is usually called the "Markov condition". One might say, loosely, that the next state of the system depends only on the present state and not earlier states.

Condition (2) means that, if the system is in state i at time k , then the probability that it will change to state j at time $k+1$ does not depend on k . We say that the transition probabilities $(p_{i,j})$ do not vary with time (k). This is called the stationarity condition. It is usually not incorporated in the definition of a Markov chain, but since we consider only stationary Markov chains, we have included (2) as part of the definition.

In probability theory, we begin by considering independent random variables, for example the results of several throws of a die. Equation (2) makes it clear that Markov chains use conditional probabilities; the random variables in X may not be independent of each other. One of the fundamental aspects of Markov chains is that they lead us into the study of sequences of dependent random variables.

These technical ideas will become clearer as one deals with the applications that follow in the next section.

Applications of Markov Chains

In this section, we present some applications that may spark the interest of Year 12 students or their teachers. Von Hilgers & Langville (2006) have published another list with a different purpose. For each application, one might ask the following questions that have been proposed by Isaacson & Madsen (1976, p. 107).

- Is the choice of states in the set S appropriate for the application?
- Is it reasonable to assume that the Markov condition stated in equation (1) above holds?
- Is it reasonable to assume that condition stated in equation (2) above holds?
- Could the transition probabilitiesbe calculated, or at least estimated, in a sensible manner?

Markov's Application

The study of Markov chains originated with a Russian mathematician, Andrei Andreyevich Markov (1856-1922). For a sketch of Markov's life, see O'Connor and Robertson (2006); Seneta (2006) gives a more detailed account. One of Markov's earliest works on the topic was presented in a lecture in St. Petersburg about a century ago in 1913 and is now available in English (Markov (2006)).

Up until Markov's time, the main emphasis in probability was the study of sequences of *independent* random variables. Such a sequence would result from tossing a coin many times and obtaining a sequence of Heads and Tails such as "HHTHTTH ..." . The outcome of each throw is independent of earlier outcomes.

Markov was interested in exploring sequences of random variables where this assumption of independence did not hold. He took some Russian text from Pushkin's poetical work "Eugeny Onegin", omitted spaces and punctuation marks, discarded two Russian letters that are not pronounced (soft-sign and hard-sign), and then classified the remaining letters as consonants or vowels. Thus the original text was reduced to a string of characters {c, v} where "c" means "consonant" and "v" means "vowel". An equivalent process in English would be to convert "The quick brown fox" to "ccvcvvccccvcccvc".

Does it appear that, in the sequence of c's and v's thus formed, there is some dependence between one character and the next? In more modern terminology, was the sequence generated by a 2-state Markov chain where the state space is {c, v}?

Such a model may have applications in linguistics, but it is unlikely that Markov had any particular application in mind. Of course, Markov did not use the term "Markov chain" but he does refer to this sequence as a "chain".

Markov's paper marks the beginning of the study of sequences of *dependent* random variables. Markov was interested in exploring probabilities such as the conditional probability that a letter was a consonant, given that the previous letter was a vowel. If the conditional probability that a letter was a consonant, given that the previous letter was a vowel, is different from the conditional probability that a letter was a consonant, given that the previous letter was a consonant, then there is dependence between successive characters in the sequence.

We have included this example mainly for its historical interest. One might say that the study of Markov chains all started with poetry! However, one aspect of Markov's work that may impress a reader of the 21st century is that he used a sample of 20,000 characters, and did all the data analysis by hand.

Health Care

Cancer is a major cause of death in Australia as in many developed nations. A great deal of clinical and scientific research has been aimed at improving our knowledge of this disease. Mathematicians have also contributed to this effort in different ways. Just over 60 years ago, Fix and Neyman (1951) modelled the progression of breast cancer as a Markov process. For simplicity, we will frame their model as a Markov chain.

We can regard a person as passing through a series of four states during a lifetime. State 1 is being alive and not being treated for cancer. State 2 is defined as being treated for cancer. State 3 is having died from cancer or the treatment (e.g., an operation). State 4 is having died from some other cause, or simply lost to the study. We might record the state of a person every six months, thus setting up the process as a discrete-time Markov chain. Let us assume that the transition probabilities obey the Markov property.

It is clear how such a model could be useful. For example, if, over several years, the transition probability of moving from State 2 (under treatment) to State 1 (alive but not under treatment) in a six month period has increased, then we have made some progress in treating cancer patients.

This is a very simple model. Fix and Neyman (1951, p. 209) offer some sage advice. "Any conceivable mathematical model of any phenomena must involve simplifications. The greater the simplifications adopted, the further one must be from the actual processes studied. On the other hand while more detailed models may (but need not) approach the phenomena satisfactorily, they may appear so complex as to lose all usefulness".

Many researchers have sought to improve the model in the way it applies to cancer. One could develop a more complex model with more states by breaking up State 2 (under treatment) into several sub-states according to the stage that the breast cancer has reached. The model could be applied for different cancers. For example, Sherlaw-Johnson et al. (1994) describe the progression of cervical cancer using a Markov chain model. Indeed, theapproach could be used to describe the progression of any chronic disease.

We end this application with a note of warning. Although this is an important, contemporary application of Markov chains, one has to be careful in using it in a classroom. Some students in the class may find this example upsetting because there is someone close to them suffering from cancer.

Snakes and Ladders

Snakes and Ladders is a popular children's game played on a board of 100 squares. We will assume that the reader is familiar with the game and we will not describe the rules in detail.

Imagine that you playing the game solo. You start with your marker off the board; we will call this state 0. Then there are 100 squares which we will call states 1, 2, … , 100. You move from state to state randomly by means of throwing a die, sometimes going up a ladder and sometimes sliding down a snake. We will not have any special rules for throwing a 6. Let us say that once you get to 100 or more then the game is over. So if your marker is on square 97 and you throw a 5 then your game is finished.

Here is a *bona fide* Markov chain. There are 101 states namely 0, 1, … , 100. The system changes from one state to another by a random process. The transition probabilities are constant during the game; thus condition (2) is satisfied. It is obvious that the Markov condition (1) holds. In other words, if you are in state i then the probability that you move to state j in the next move depends only on i and j and not on the states in any earlier stage of the game. We say that the game does not "remember" earlier parts of the game.

By contrast consider the game of chess.Although chess does not involve randomness, it has anon-Markovian aspect. For example, you cannot castle if you have moved the King or Rook in some earlier stage of the game. Thus, in chess, earlier moves can have an impact on the next move.

Snakes and Ladders is a useful example for illustrating the basic concepts of Markov chains in the classroom. School text books focus on 2-state Markov chains for simplicity. Snakes and Ladders is an example of a Markov chain that is readily understood even though it has 101 states, and meets the definition of a Markov chain exactly. Other authors have written about this game as a Markov chain; see, for example,Johnson (2003) and Cheteyan et al. (2011); note that, in the US, "Snakes and Ladders" is known as "Chutes and Ladders".

Searching the WWW

Searching the World Wide Web is part of everyday life in 2012. In the early days of the WWW there were several competing search engines. But then Google emerged as the leader. One reason for the success of Google is the algorithm that underpins its search engine. The algorithm is known as PageRank and it is an application of linear algebra and Markov chains. An account of the history behind this development and some technical details can be found in Langville & Meyer (2006). In this section, we explain the connection between this search algorithm and Markov chains; our explanation is based on Langville & Meyer (2006).

Suppose that there are m pages on the WWW; obviously this is a very large number. We can regard surfing the WWW as a realization of a Markov chain. A surfer starts at one page, say Page 1, and then jumps to another page, say Page 2, by following the links on Page 1. If we assume that the surfer chooses a link at random from the links on Page 1, then we can assign transition probabilities to the possible jumps from Page 1. The size of this $m \times m$ matrix is huge. However, most of the entries are zero because any given page is linked to only a small number of other pages. We say that the matrix is sparse.

Although Sergey Brin and Larry Page, the creators of the PageRank algorithm, did not mention Markov chains in their original papers, these days the method is viewed as an application of a Markov chain. Further details can be found in Langville & Meyer (2006).

Marketing

Mathematical models play an important role in business analysis. In this section we present an example of the application of Markov chains to a problem in marketing research which was inspired from reading Deming and Glasser (1968).

A telephone company offers its customers several plans for their mobile phones. The company knows the distribution of its customers currently across all the plans and it is interested in forecasting the distribution in the next two years. The Markov chain may be a suitable model for the setting of this problem.

The set of states would be the set of plans. The plans may include "pre-paid" or "cancelled" as well as 6-month plans and 12-month plans. The state of each customer might be recorded every month.

There may be rules that prevent transferring between certain plans; in such cases the transition probability would be zero. Other transition probabilities could be estimated from recent data.

The distribution of customers across various plans could be determined from the initial distribution and the matrix of transition probabilities. Since this industry changes rapidly, one might be safe in using this approach for short-term forecasting.

We recommend the paper by Deming & Glasser (1968) because it has been written with considerable clarity.

Conclusions

We have presented a brief introduction to Markov chains to assist teachers who have had little experience with them.

"What is this good for?" is a common question posed by students studying mathematics. The applications of Markov chains presented in this paper can be used to answer this question. Many applications of these models involve a large number of states. However, the basic ideas involved in conceiving a Markov model to describe a game like Snakes and Ladders or searching the WWW are simple in spite of the large number of states.We hope that our selection will enhance discussion of applications in the classroom.

An important theme in the mathematical subjects in the Australian curriculum is to make connections between mathematics and the wider world. The study of Markov chains can assist in making such connections.

References

Cheteyan, L., Hengeveld, S., & Jones, M.(2011). Chutes and ladders for the impatient. *The College Mathematics Journal, 42*(1), 2-8.

Coffey, D., Boucher, K., Matheson, A., Kolsch, P., Anastasio, R., Jaskula, I., & Howard, J. (2009). *Mathematical methods 3 & 4: CAS enhanced.* Port Melbourne: Pearson Aust.

Deming, W., &Glasser, G. (1968).A Markovian analysis of the life of newspaper subscriptions. *Management Science, 14*(6),B283-B293.

Fix, E., & Neyman, J. (1951). A simple stochastic model of recovery, relapse, death and loss of patients. *Human Biology, 23*(3), 205-241.

Isaacson, D., & Madsen, R. (1976). *Markov chains: Theory and applications.* New York: John Wiley & Sons.

Johnson, R. (2003). Using games to teach Markov chains. *Primus: Problems, resources, and issues in mathematics undergraduate studies, 13*(4), 337-348.

Langville, A., & Meyer, C. (2006). *Google's PageRank and beyond: The science of search engine rankings.* Princeton: Princeton University Press.

Markov, A. (2006). An example of statistical investigation of the text Eugene Onegin concerning the connection of samples in chains. *Science in Context, 19*(4), 591–600.

O'Connor, J., & Robertson, E. (2006). Andrei Andreyevich Markov. Retrieved from http://www-history.mcs.st-andrews.ac.uk/Biographies/Markov.html.

Seneta, E. (2006). Markov and the creation of Markov chains. In A. Langville, A., & W. Stewart (Eds.), *MAM 2006: Markov Anniversary Meeting. An international conference to celebrate the 150th Anniversary of the birth of A.A. Markov* (pp. 1-20).Raleigh: Boson Books.

Sherlaw-Johnson, C., Gallivan, S., Jenkins, D., & Jones, M. (1994).Cytological screening and management of abnormalities in prevention of cervical cancer:an overview with stochastic modeling.*Journal of Clinical Pathology, 47*(5), 430-435.

Von Hilgers, P., & Langville, A. (2006).The five greatest applications of Markov chains. In A. Langville & W. Stewart (Eds.),*MAM 2006: Markov Anniversary Meeting. An international conference to celebrate the 150th Anniversary of the birth of A.A. Markov* (pp. 155-168).Raleigh: Boson Books.

USING *MATHEMATICA* IN THE CLASSROOM – BY TEACHERS FOR TEACHERS

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Mathematica is a computational software package that can be used as a powerful teaching and learning tool. It helps demonstrate concepts, creates support materials, assessment tasks, and presentations, and engages students in interactive learning, exploring and developing an understanding of mathematical concepts. Apart from basic and more complex calculations similar to the features of other CAS tools, Mathematica can be used to produce animations and manipulations, slide shows, interactive tests and other live documents. A free trial version of Mathematica can be downloaded from Wolfram Research(1).

Starting on *Mathematica*

The teaching and learning of mathematics is becoming increasingly challenging with the expectation that classroom teachers provide ever more engaging lessons and cater for the diverse learning styles of students. In the search for tools to support the understanding of mathematical concepts and knowledge, *Mathematica* has proven to be very powerful for both the teaching and learning processes. *Mathematica* has four main features that are required for correct syntax:

1. There are no spaces between characters.
2. Most commands start with a capital letter. If a command is made up of two or more words, each word starts with a capital letter, eg. PlotStyle
3. Commas are used to separate arguments.
4. To evaluate a cell always press Shift + Enter.

Lesson Starters Years 7 – 10

In this section there is a selection of three tasks at the junior level, years 7-10. They can be used as they are or can be adapted to suit the needs of the reader.

Prime Numbers Investigation

This investigation is suitable for year 7 students to explore the nature of prime and composite numbers, and the concept of factors.

Example 1

a Find all prime factors of 9.

Input line: **`FactorInteger[9]`** ... gives all the prime factors of 9

Output line: **`{{3,2}}`**, where 3 is the prime factor and 2 is the power.

```
In[1]:= FactorInteger[9]
Out[1]= {{3, 2}}
```

Figure 1. Screen shot of input and output lines

b Factorise all integers from 2 to 11 in table form

Input line: **`Grid[Table[{1+n,FactorInteger[1+n]},{n,1,10}]]`**

... arranges all the factors of the integers from 2 to 11 in a table.

```
In[2]:= Grid[Table[{1 + n, FactorInteger[1 + n]}, {n, 1, 10}]]
          2      {{2, 1}}
          3      {{3, 1}}
          4      {{2, 2}}
          5      {{5, 1}}
Out[2]=   6  {{2, 1}, {3, 1}}
          7      {{7, 1}}
          8      {{2, 3}}
          9      {{3, 2}}
          10 {{2, 1}, {5, 1}}
          11    {{11, 1}}
```

Figure 2. Screen shot of input and output lines

c Find prime numbers.

<u>Input line:</u> **`Table[Prime[n],{n,50}]`**

… lists the first 50 prime numbers.

```
In[3]:= Table[Prime[n], {n, 50}]

Out[3]= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67,
  71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149,
  151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229}
```

Figure 3. Screen shot of input and output lines

This investigation can be extended to suit the purpose of the lesson and the group of students taught.

Introduction to the Sum of the Interior Angles of a Triangle

This task was designed as a teaching lesson to prove that the sum of the interior angles of a triangle is 180°. It is suitable for year 7 or 8 students.

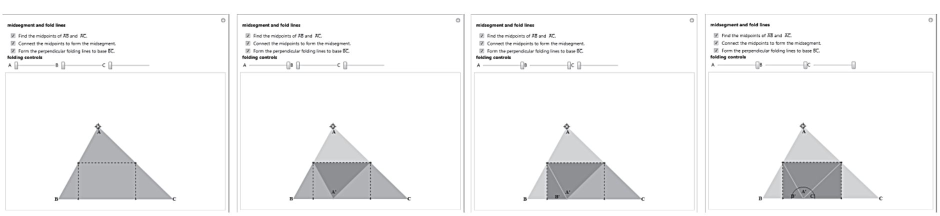


Figure 4. Screen shots for the demonstration: The Sum of the Interior Angles of a Triangle Equals 180 Degrees by Paper Folding (HREF1)

In the second part of this lesson Mathematica is used to solve linear equations involving finding an unknown angle given the magnitudes of the other two angles.

Example 2

In a triangle ABC, ∠ B = 46°, ∠ C = 75°. Find the magnitude of ∠ A.

The *Mathematica* command used to solve the equation A + 48° + 27° = 180° is:

<u>Input line:</u> **`Solve[a+48+27==180,a]`** … notice the use of double equals signs instead of a single equals sign. The use of "**`,a`**" is to indicate the variable of the equation.

```
In[4]:= Solve[a + 46 + 75 == 180, a]

Out[4]= {{a → 59}}
```

Figure 5. Screen shot of input and output lines

Statistics Summary

This task was designed as a revision of the statistical concepts learnt for years 9 or 10 students as required.

```
In[5]:= StemLeafPlot[{1.2, 2.5, 4.1, 1.6, 3.8, 2.6, 2.9}]

        Stem | Leaves
           1 | 26
           2 | 569
Out[5]=    3 | 8
           4 | 1

        Stem units: 1
```

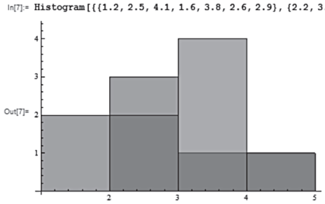
```
In[6]:= StemLeafPlot[{1.2, 2.5, 4.1, 1.6, 3.8}, {2.3, 2.8, 1.4, 3.6}]

        Leaves | Stem | Leaves
            62 |  1   | 4
             5 |  2   | 38
Out[6]=      8 |  3   | 6
             1 |  4   |

        Stem units: 1
```

Figure 6. Screen shots of input and output lines for basic stem and leaf plots and back-to-back stem and leaf plots

```
In[7]:= Histogram[{{1.2, 2.5, 4.1, 1.6, 3.8, 2.6, 2.9}, {2.2, 3.5, 3.1, 2.6, 4.8, 3.6, 3.9}}]
```



```
In[8]:= Histogram3D[{{1.2, 1.2}, {2.5, 2.5}, {4.1, 4.1}, {1.6, 1.6}, {3.8, 3.8}, {2.6, 2.6}, {2.9, 2.9}}]
```

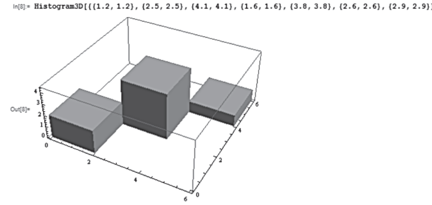


Figure 7. Screen shots of input and output lines for a 2D representation of a histogram and a 3D representation of the histogram

```
In[9]:= Mean[{1, 7, 3, 10, 5, 2}]

         14
Out[9]= ----
          3
```

```
In[10]:= Quantile[{1.2, 2.5, 4.1, 1.6, 3.8, 2.6, 2.9}, {1/4, 1/2, 3/4}]

Out[10]= {1.6, 2.6, 3.8}
```

Figure 8. Screen shots of input and output lines used to evaluate the mean, median and quartiles

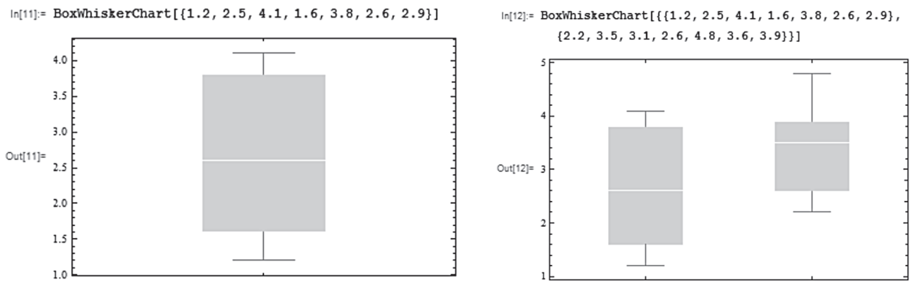


Figure 9. Screen shots of input and output used for box and whisker plots

Lesson Starters Years 10 – 12

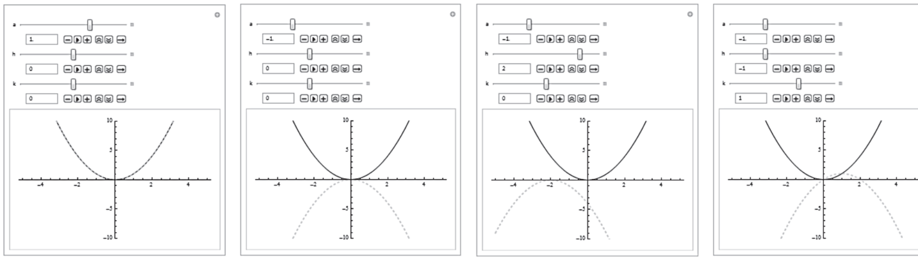
In this section there is a selection of four tasks at the senior level, years 10-12. They can be used as they are or can be adapted to suit the needs of the reader.

Looking at Parabolas Demonstration

This task was designed as a teaching-demonstration lesson. This lesson is suited to years 9, 10 or 11 as required. The manipulation constructed in *Mathematica*, looks at the behaviour of quadratic functions of the form $g(x) = a(x + h)^2 + k$, investigating the effect of the constants a, h, and k on the shape of the function and its position in the Cartesian plane relative to the function $f(x) = x^2$.

The commands used in this manipulation are becoming more sophisticated with the introduction of the range of the function, PlotRange→{-10,10}, and different colours for the two graphs, Black, for $f(x) = x^2$ and {Green,Thick,Dashed} for the function $g(x) = a(x + h)^2 + k$.

```
Manipulate[Plot[{x^2,a(x+h)^2+k},{x,-5,5},PlotRange→{-10,10}
    PlotStyle→{Black,{Green,Thick,Dashed}}],
    {a,-3,3,0.5},{h,-3,3,1},{k,-3,3,1}]
```



$a = 1, h = 0, k = 0$ $a = -1, h = 0, k = 0$ $a = -1, h = 2, k = 0$ $a = -1, h = -1, k = 1$

Figure 10. Screen shots of output lines for the manipulation of $g(x) = a(x + h)2 + k$

Each of the constants *a, h,* and *k* can be manipulated individually or concurrently. Due to its visual and interactive features, this manipulation provides a suitable introduction to transformations of functions.

Average and Instantaneous Rates of Change

This task was designed as a teaching-demonstration lesson for units 1 and 2 Mathematical Methods, providing interactive visualisation of the average and instantaneous rate of change. It can also be used as revision in units 3 and 4 Mathematical Methods and introduces the Slide show feature of *Mathematica*.

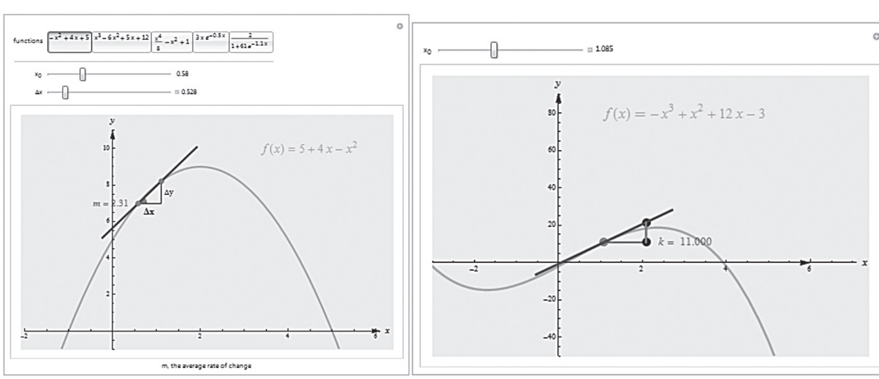


Figure 11. Screen shots for the demonstrations: Average Rate of Change Exploring More Functions (HREF2) and demonstrations: Instantaneous Rate of Change (HREF3)

Integral Calculus

This task was designed as a summary lesson of Integral Calculus and the corresponding *Mathematica* commands required for Units 3 and 4 Mathematical Methods students.

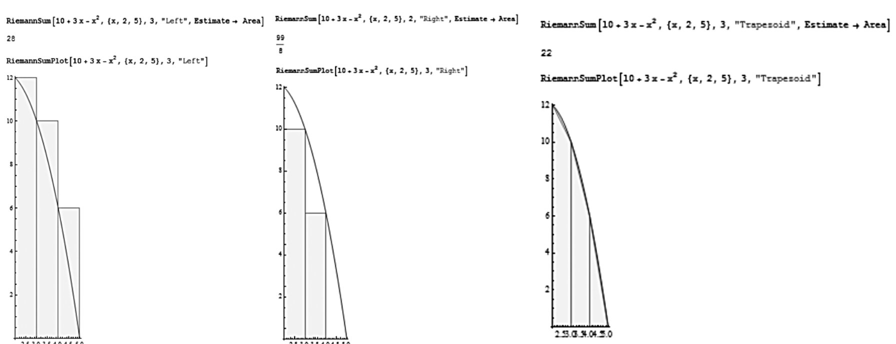
The three methods of finding the area under a curve using approximations are shown in Figure 11. The rectangle approximation using the "leftend point estimate" is calculated using the command:

`RiemannSum[function,{x,xmin,xmax},n,"Left",Estimate->Area]`, where **`n`** represents the number of rectangles. Similarly, the rectangle approximation using the "rightend point estimate" is calculated using the command:

`RiemannSum[function,{x,xmin,xmax},n,"Right",Estimate->Area]`, where **`n`** represents the number of rectangles.

The third estimate is the trapezium estimate with the command:

`RiemannSum[function,{x,xmin,xmax},n,"Trapezoid",Estimate->Area]`, where **`n`** represents the number of strips.



a **b** **c**

*Figure 12. Screen shots of input and output lines for the **a** "leftend point estimate", **b** "rightend point estimate", **c** "trapezium estimate"*

Basic integration is being evaluated using the **`Classroom Assistant ->Calculator->Advanced->\[Integral](function)dvariable`**

In[1]:= $\int (a\,x + b)^n\,dx$ Out[1]= $\frac{(b + a\,x)^{1+n}}{a + a\,n}$

In[2]:= $\int (a\,x + b)^{-1}\,dx$ Out[2]= $\frac{\text{Log}[b + a\,x]}{a}$

In[3]:= $\int (e^{k\,x})\,dx$ Out[3]= $\frac{e^{k\,x}}{k}$

In[4]:= $\int \text{Sin}[a\,x]\,dx$ Out[4]= $-\frac{\text{Cos}[a\,x]}{a}$

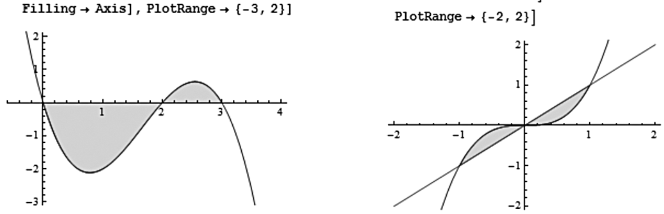
Figure 13. Screen shots of input and output lines for integrals of various functions

To represent the area under a curve and the area between two curves, the following commands are used:
Show[Plot[function,{x,x$_{min}$,x$_{max}$}],Plot[function,{x,x$_{lowerlimit}$,x$_{upperlimit}$ },Filling→Axis],PlotRange→{y$_{min}$,y$_{max}$}] and

Show[Plot[{function1,function2},{x,x$_{min}$,x$_{max}$}],Plot[{function1, function2},{x,x$_{min}$,x$_{max}$},Filling→{1→{2}}],PlotRange→{y$_{min}$,y$_{max}$}]

```
Show[Plot[{x (2 - x) (x - 3)}, {x, -1, 4}],
 Plot[{x (2 - x) (x - 3)}, {x, 0, 3},
  Filling → Axis], PlotRange → {-3, 2}]
```

```
Show[Plot[{x^3, x}, {x, -2, 2}],
 Plot[{x^3, x}, {x, -1, 1},
  Filling → {1 → {2}}],
 PlotRange → {-2, 2}]
```



In[5]:= $-\int_0^2 x\,(2 - x)\,(x - 3)\,dx + \int_2^3 x\,(2 - x)\,(x - 3)\,dx$

Out[5]= $\frac{37}{12}$

In[6]:= $\int_{-1}^0 (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx$

Out[6]= $\frac{1}{2}$

Figure 14. Screen shots of input and output lines for integrals of various functions

Equations of Circles

This task was designed as a teaching, learning and practice lesson in which students are required to complete a series of exercises after an introduction to the equation of a circle in standard and non-standard forms. It also uses the Slide show feature of *Mathematica* and grouping of cells.

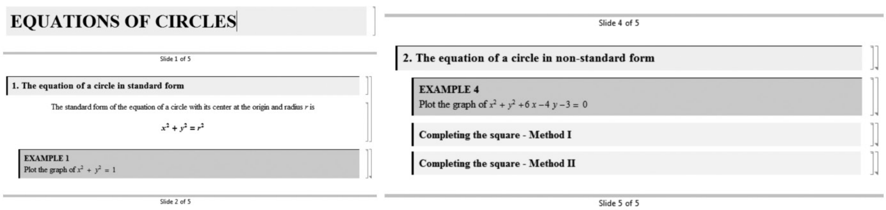


Figure 15. Screen shots with text, slides and cell groupings

The end of lesson exercises are designed to provide feedback and practice of the topic taught and can be used as a form of assessment as learning.

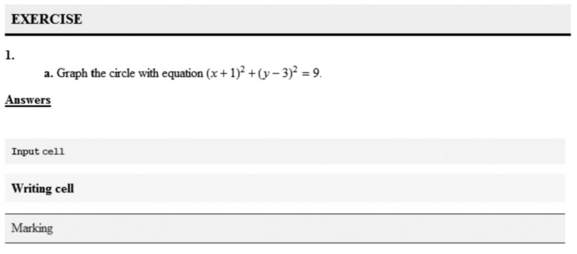


Figure 16. Screen shot of part of the Exercises

Conclusion

Mathematica is a powerful computational tool which allows the user to produce quality, interactive and visual classroom activities covering a variety of tasks; from teaching-demonstration type of lessons, to investigations, slide shows, interactive manipulations, interactive assessment tasks and much more. Once started on *Mathematica*, the wide range of resources available makes the teaching and learning of Mathematics stimulating and engaging.

References

Computer software

Wolfram, S. Mathematica (Version 8) [Computer software]. Wolfram Research

Websites

HREF1: Wolfram Demonstrations Project. *The Sum of the Interior Angles of a Triangle Equals 180 Degrees by Paper Folding* contributed by Corcoran, Sean (Virginia Tech). Demonstration retrieved from http://demonstrations.wolfram.com/TheSumOfTheInteriorAnglesOfATriangle Equals180DegreesByPaperF/

HREF2: Wolfram Demonstrations Project. *Average Rate Of Change Exploring More Functions* contributed by Wolfgang Narrath and Reinhard Simonovits. Demonstration retrieved from http://demonstrations.wolfram.com/AverageRateOfChangeExploringMoreFunctions/

HREF3: Wolfram Demonstrations Project. *Instantaneous Rate Of Change* contributed by Wolfgang Narrath and Reinhard Simonovits. Demonstration retrieved from http://demonstrations.wolfram.com/InstantaneousRateOfChange/

INNOVATIVE TEACHING WITH TECHNOLOGY IN THE LIGHT OF THE THEORY OF DISTRIBUTED CONGNITION

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This paper addresses 'Technological Pedagogical And Content Knowledge' (TPACK) the specialized knowledge that teachers must acquire in order to use technology in their instructions innovatively. The impact of technology on the students learning process is then discussed based on the theory of distributed cognition. Examples to illustrate the application of technological pedagogical and content knowledge with the use of the graphing calculator and dynamic geometry software to create innovative instruction are also provided in the paper.

Introduction

'Innovation' is the buzzword of our times (Lessem & Schieffer, 2010). Innovation is defined by Gallo (2011) author of the book, The innovation Secrets of STEVE JOBS as "a new way of doing things that results in positive changes". Translated to the pedagogical arena, this implies new ways of teaching that results in positive changes in the learning process and leaning outcome of the students.

Today the enhancement in educational technology provides every teacher a powerful means to approach old content in a new way that can result in positive pedagogical changes. However, technology on its own is not capable of delivering innovative lessons that can result in positive changes in students' learning. Technology itself does and will not impact students' learning. The power of technology lies in the intersection of technology, pedagogy and content (Greenhow, 2009) - the power of technology lies in the teacher. As Shulman (1986) stated teachers have specialized knowledge that sets them apart from other professions. Shulman explained that a brilliant mathematician would not necessarily make an excellent teacher. He argued that an excellent teacher has special knowledge which lies at the intersection of content and pedagogy and hence he called this special knowledge Pedagogical Content Knowledge.

Similarly, Mishra and Koehler (2009) claimed that quality teaching is not a process of copying a few instructional techniques. Instead it emerges from deep thinking of the teacher in conjunction with the discipline that has to be taught making content intellectually accessible to the students. Integrating technology adds a new knowledge and hence Shulman's framework has been updated and the special knowledge when using technology is called 'Technological Pedagogical And Content Knowledge (TPACK)' (see framework in Figure 1) (Mishra & Koehler, 2009).

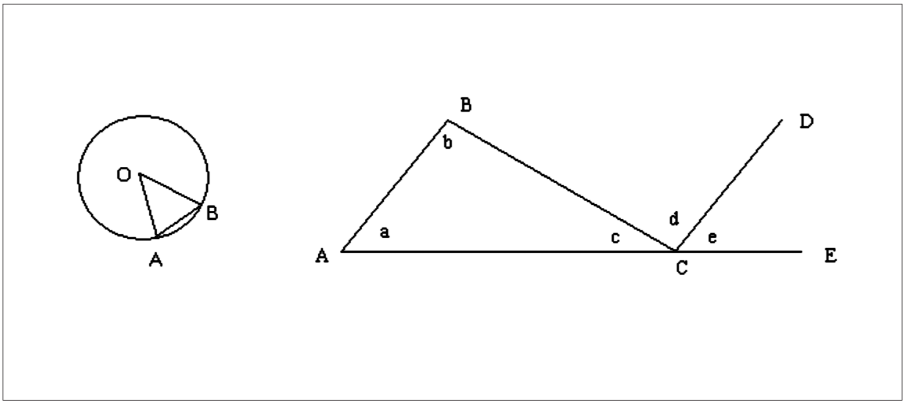

Figure 1: TPACK Framework

Today mathematics teachers can be innovative teachers by using computer technology (referred in this paper as technology). However, the impact of their innovative mathematics lesson with the use of technology in the teaching and learning sessions will depend on their Technological Pedagogical and Content Knowledge.

Many researches in the past have shown that the use of technology has had positive effects on students learning (Obara, 2010; Herman & Laumakis, 2009; Nemirovsky, 1994; Ainley, 1994; Thornton & Sokoloff, 1990; Mokros & Tinker, 1987; Brasell, 1987). To understand the success of learning with the use of modern technology, I turn to the theory of distributed cognition

Distributed Cognition and the use of Technology

The theory of distributed cognition was developed by Ed Hutchins and his colleagues at University California, San Diego in the mid to late 80s (Rogers & Scaife, 1997). This theory claims that cognition is better understood as a distributed phenomenon, in contrast to the traditional view of cognition as a localised phenomenon that is best explained in terms of information processing at the level of the individual.

Salomon, Perkins and Globerson (1992) adopting this phenomenon summarised cognitive effects when using technology as "effects with technology obtained during intellectual partnership with it, and effect of it in terms of the transferable cognitive residue that this partnership leaves behind in the form of better mastery of skills and strategies."

To explain distributed cognition as "effects with technology", I turn to Döfler's (1993) view of cognitive processes (which also adopts Hutchins' view of cognition). Döfler (1993) suggested that cognitive processes be viewed as a system made up of the individual, the whole context and the multiple relationships between them. Thus, the cognitive system has the subject (the individual) and the available cognitive tools which would aid the thinking process. Cognitive tools can be paper and pencil, calculators, computers, graphing calculators, television, etc. Döfler (1993) compared the thinking process to doing physical work stating that 'There is no such thing as "pure" work without using any tool'. To attain the specific goals, one has to use tools in an appropriate organized manner. To illustrate the thinking process as a system, Döfler (1993) used the artist as an analogy. Döfler stated:

> The skill and the intelligence of an artist like a painter are more appropriately viewed as being realized by the whole system consisting of the human individual and all his tools. These tools do not just express ideas and imaginations pre-existing in the mind of the artist and independently of the tools. Rather, the system of painter, brush, colours, canvas, etc. realizes the painting (Döfler,1993, p. 173)

The thinking process can be explained in terms of 'distributed cognition'. Distributed cognition refers to the earlier described 'system' – the individual and the available tools where cognition is viewed as distributed over them (Döfler, 1993). According to this view of thinking, to solve a given mathematical problem, the individual can employ the available tool and his or her own mind to solve it.

To explain distributed cognition in mathematics, let us take for example the drawing of a straight line of a specific length, for example 5 cm in length. Using a ruler one can produce the 5 cm straight line. The individual need not have the skill of drawing a straight line unaided (that is free hand), nor does the individual need a mental representation of 5 cm. If the use of a ruler is not permitted, then fewer people will be able to draw a 5 cm straight line if they lack either the skill to draw a straight line or the appropriate mental representation of a length of 5 cm. With the use of a ruler, cognition is distributed in the process for producing a 5 cm straight line – the mental representation of 5 cm and the skill to produce a line which is straight are taken over by the ruler and the individual has to have knowledge about how to use a ruler to produce the 5 cm straight line.

To explain the "effects of technology", I turn to Pea's (1985) view that 'intelligent' technology "offloads" part of the cognitive process as a result of distribution of cognition, allowing the user to focus cognitive resources elsewhere. Pea claims that over time the user will develop cognitive skills to accomplish many of the cognitive processes demonstrated when using technology and would be capable of demonstrating these skills without any longer requiring the aid of the technology.

Let us refer to identifying acute, obtuse and right angles based on the respective definition of each type of angle. Using a protractor enables an individual to measure a given angle between two lines and identify the category of the angle. Since the protractor provides the measure of the angle -'intelligent' technology "offloading" part of the cognitive process as a result of distribution of cognition, the user is allowed to focus cognitive resources elsewhere - the individual can observe the appearance of the various angles. Later the individual would be capable of recognizing the different types of angles without any longer requiring the aid of the protractor because he/she has developed the cognitive skill to identify the category of the angle.

Similarly using technology to draw graphs or geometric objects or perform other mathematical tasks affords students more time to focus on the actual objective of the learning session. This then would enable students to expend their cognitive processes to develop the desired skills and concepts of the lesson.

Let us examine some examples of how the distribution of cognition using appropriate TPACK enabled me to create innovative mathematics lessons which transformed the content in a way that made it intellectually accessible to the students and enabled them to acquire the desired skills and concepts.

Teaching and Learning Mathematics with Technology

This section describes specific teaching and learning activities which have been used and proved to be successful, to illustrate how the distribution of cognition with the application of Technological Pedagogical and Content Knowledge provided to create innovative lessons.

Using the TI-84 Plus Graphing Calculator to Develop Students' Understanding of Scales

In the Malaysian mathematics curriculum, 'graphs of functions', is a topic learnt in the secondary level (students aged 16-17). Graphs however are drawn on paper and the scales to construct graphs illustrating the covariance of two variables are often given and students mechanically use the given scales and complete the construction of the graphs. To change scales and redraw a graph is time consuming and hence how a scale should be adjusted to fit a graph or diagram is poorly understood by students even after completing the topic. Using technology certainly made a difference among my Bachelor of Education students to explore scales in a fun manner. They explored scales when graphing parametric trigonometric functions which created interesting shapes.

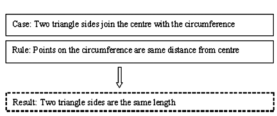

Figure 2: Parametric equations

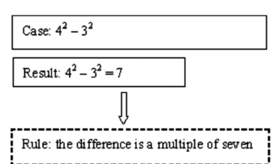

Figure 3: Graph of Parametric Equations

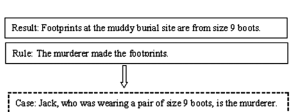

Figure 4: Window Setting for Figure 5

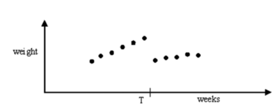

Figure 5: Graph based on Figure 4 setting

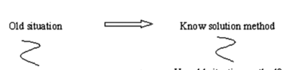

Figure 6: Window Setting for Figure 7

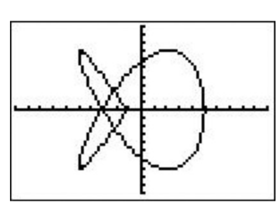

Figure 7: Graph based on Figure 6 setting

The activity started by providing each student a graphing calculator and showing Figure 2 followed by Figure 3 projected on a screen in class. Then the students were asked to guess the shape displayed. All the students claimed that the graph was too small to make out the shape. This then led to request for suggestions on how to make the graph bigger. The first response was 'zoom-in', but when they tried it, it proved to be futile. Then one student suggested changing the scale. Hence I taught them to use the window settings. Students then explored various scales and had discussions as to how to produce a bigger fish. The activity became more interesting to the students when they saw a fish instead of a boring line graph. The students had fun trying many different scales and very quickly grasped how to make the graph bigger or smaller. They also learnt to change the range so that the graph was in the centre of the screen. The power of the technology is obviously due to the distribution of cognition over the graphing calculator and the student. While the graphing calculator took over the cognitive process of producing and positioning the graph according to the scales and range keyed in, the students focused on how to adjust the scale to obtain a maximum size graph to fit the screen and to adjust the range to centre the graph on the screen.

Giving students then a graph on paper which occupied a small corner of the graph paper immediately produced responses commenting on the inappropriateness of the scale and range and intelligent suggestions were made confidently and accurately to reconstruct the graph to fit the entire page and position the graph more centrally. Hence Pea's (1985) claims that over time the user will develop cognitive skills to accomplish many of the cognitive processes demonstrated when using technology and would be capable of demonstrating these skills without any longer requiring the aid of the technology proved to be valid.

In the above lesson the interest of the pupils was drawn because the content was transformed in a way that made it intellectually accessible to the students, that is, effective Technological Pedagogical and Content Knowledge had been employed to conduct an innovative lesson which had a great impact on pupils' learning of scales. The meaningful understanding of the knowledge that the students had acquired when using the technology enabled them to apply it even without the use of technology. If the students had been merely provided with the graphing calculator and asked to construct line graphs according to structured instructions as provided in the graphing calculator guide book on how to draw graphs, the lesson would replicate the traditional instructional practice and would have made no impact on the learning process. Hence the effectiveness of the use of the technology lies in the teacher's Technological Pedagogical and Content Knowledge and not solely on the technology.

Using the CBR to Understand Distance-Time Graphs

This activity focused on the use of the Calculator Based Ranger (CBR) a data collecting device to enhance lower secondary students understanding of distance - time graphs. The first part of the lesson allowed students to explore distance - time graphs using the CBR. Students were instructed to walk at different speeds and observe the graphs displayed on the screen. This enabled them to deduce that the steeper the slope the greater the speed. They also realised that a horizontal line represented no change in distance from the ranger and a negative gradient graph indicated starting away from the CBR.

The second part of the lesson involved interpreting and re-creating given graphs. Students were challenged to produce graphs like the ones given in Figure 8 using the CBR. The process of discussion to produce the graphs demanded a sound understanding of the distance-time graph. Students were soon able to reason out how to produce the challenging graphs and also why some of the graphs were impossible to produce.

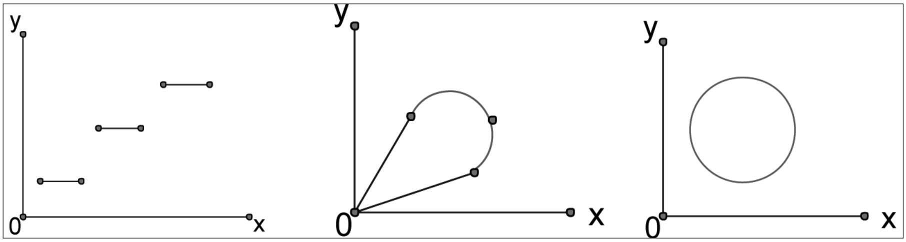

Figure 8: Challenging and Impossible Graphs

The final activity of the lesson challenged students to produce a horizontal graph as shown in Figure 9, with a person moving.

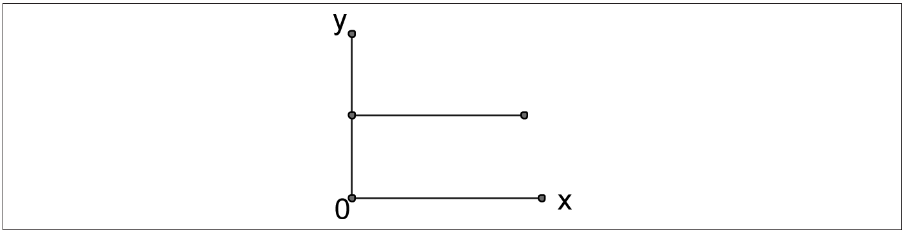

Figure 9: Horizontal Graph

Their experience when trying to produce a circular graph in the earlier activity enlightened them on producing the graph by rotating the CBR and walking in a circle to produce a horizontal line graph with the person moving. Another group of brilliant students went on to tie the CBR at one end of a metre rule and position one group member at the other end. They walked the group member about the class to produce a horizontal graph. They explained that as long as the distance between the CBR and the group member was constant, the graph would be a horizontal line because time was passing but there was no change in distance.

Again in this activity, the distribution of cognition between the technology and the students enabled the students to focus on understanding the distance-time graphs, rather than been bogged down to merely constructing the graphs from meaningless tables of values. Hence technology provided a means for a teacher to present the content in a way that made it intellectually accessible to the students and hence produced an innovative lesson for the understanding of distance-time graphs. Again application of effective Technological Pedagogical and Content Knowledge to design instruction created an innovative lesson which equipped students with knowledge which they would be able to apply even without the technology, that is, to be able to interpret distance-time graphs and determine which distance-time graphs are possible and why.

Using Dynamic Geometry Software to Understand the Formula for Area of a Triangle

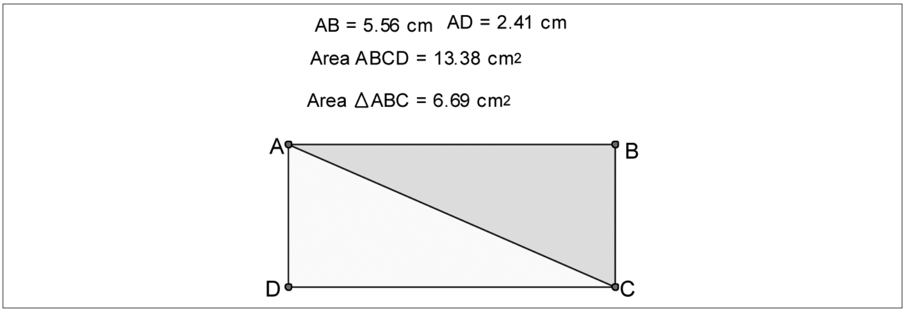

Figure 10: Dynamic geometry software to obtain the formula for the area of a triangle.

This activity started with a rectangle displayed on the screen. The rectangle was drawn using the software. The students who were lower secondary school students were requested to recall the formula for area of a rectangle that they had learnt in the primary school. The software was used to measure the length and breadth of the rectangle and the students were requested to calculate the area using their formula. Their answers were then checked using the software which displayed the area in square centimetres as shown in Figure 10. Then dividing the rectangle into two equal parts produced two right angled triangles. Students were asked to suggest a formula for each of the triangles and explain their formula. They reasoned that the area of each triangle was half the area of the rectangle. Hence the area of the triangle formula was obtained. To further investigate if their formula for the area of triangle was correct, they were instructed to drag the sides of the rectangle and record the changing values of the dimensions (length, breadth and area) of the rectangle and triangle. They then calculated the area of the respective triangles from the values of the length and breadth using their formula for the area of triangle and compared their worked out answers with the data displayed . This assured the students that their formula for area of triangle was correct.

The next part of the activity was to help students to realise that the area of a triangle remains unchanged as long as the height and base remains unchanged and that this is irrespective of the shape of the triangle. For this the diagram in Figure 11 was created. Point A moves on the line PQ and the two line segments PQ and RS are parallel. The perpendicular distance AD is the height of the triangle. The students realised that as long as the height and the length of the base remained unchanged the area remained unchanged although the perimeter changed with the shape of the triangle. They were then asked to identify the height of the triangle if the base was AC or AB. Using the software the appropriate perpendicular lines were dropped from the respective vertices and the appropriate heights and bases were measured. The area of the triangle was then calculated using the formula. The objective of the activity was achieved.

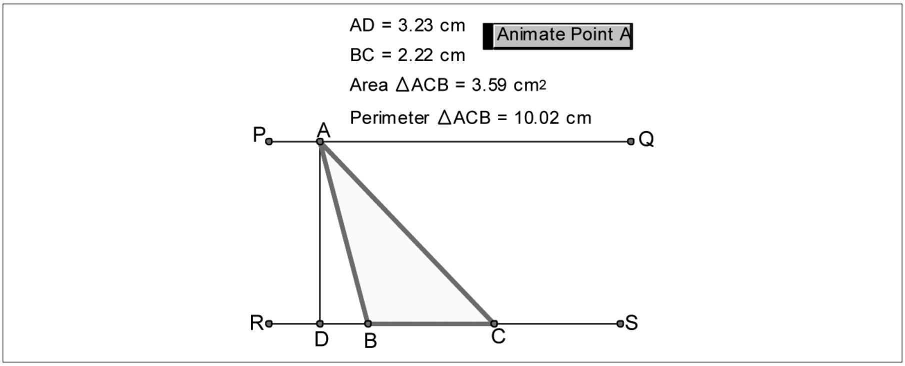

Figure 11: Dynamic geometry software to explore area of triangle formula

Finally students were asked questions on paper to find the area of given triangles and all the students were able to answer all the questions without any difficulty.

The technology sped up the exploration process by taking over the process of drawing and measuring the triangles with a ruler. Hence, the distribution of cognition between the technology and the students enabled the students to focus on discovering and understanding the formula of the area of a triangle rather than been told the formula or merely cutting out one triangle and assuming that the formula fits all triangles. They were also able to explore and realise that the area of a triangle can be calculated using any of the sides as the base as long as the corresponding height was substituted in the formula. Again the use of technology equipped students with knowledge that could be applied even without the technology. The activities using technology enabled the creation of an innovative lesson for discovering and understanding the formula for area of a triangle by the use of effective Technological Pedagogical and Content Knowledge.

Using Dynamic Geometry Software to Demonstrate Application of the Isometries of the Plane in a Creative Manner

In the teachers' training institutes in Malaysia, students are required to demonstrate application of the isometries of the plane in a creative manner by producing an Escher type tessellation in their Geometry course. When students use the paper medium they produce simple uninteresting geometrical figures because of the difficulty to cut paper shapes accurately and paste them as desired. The process to produce an Escher type tessellation can also be time consuming in the paper medium.

Teaching and learning isometries of the plane using a dynamic geometry software equipped the students with a tool that made it convenient for them to demonstrate their creativity when applying the isometries of the plane to produce an Escher type tesselation. Figure 12 shows an example of a motif which was created from a square using translation.

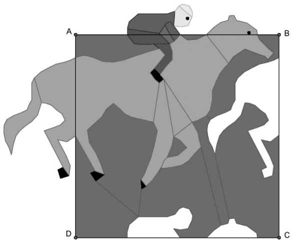

Figure 12: Motif created from a square using translation.

The motif in Figure 12 was then translated using appropriate vectors to cover the plane (see Figure 13).

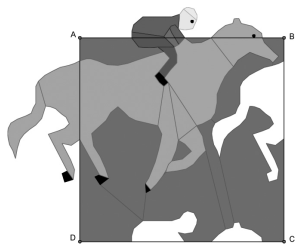

Figure 13: Tessellation created from the motif in Figure 9

In this activity the teacher shared her technological and content knowledge through her Technological Pedagogical and Content Knowledge with her students. The students having acquired the skill to use the technology to perform the isometries of the plane were able to apply these skills and concepts to be creative. The students' ideas were expressed clearly because the thinking process of the student was aided by the available cognitive tool (the dynamic geometry software).

Hence creativity was possible because of the distribution of cognition over the student and the available tool. Technology took over a large part of the cognitive process which involved menial steps (such as drawing, cutting and pasting shapes). This afforded the students more time to focus on applying the learnt mathematical knowledge, in this case the isometries of the plane.

In this example, TPACK of the teacher equipped the students with both, the content and technological knowledge. Hence this activity in addition to presenting the content in an intelligently accessible way, also educated the students to use technology to apply the knowledge acquired in a creative manner.

Conclusion

The use of technology to produce innovative instructions can be demonstrated. However, the effectiveness of the technology lies in the teacher's Technological Pedagogical and Content Knowledge. For a teacher to be able to use technology effectively in lessons, he or she must have the required technological knowledge. Then using the acquired technological knowledge in collaboration with pedagogical content knowledge will determine the extent to which the content of the lesson is presented in an intellectually accessible manner to the students and then and only then can the technology have a great impact on the students learning process.

References

Ainley, J. (1994). *Building on children's intituions about line graphs.* Proceedings of the 18th PME Conference, Lisbon, Portugal, Vol. II, pp. 1-8.

Brasell, H. (1987). *The effects of real-time laboratory graphing on learning graphic representations of distance and velocity.* Journal of Research in Science Teaching, 24(4), 385–395.

Döfler, W. (1993). *Computer use and views of the mind.* In C. Keitel & K. Ruthven (Eds.), Learning from computers: Mathematics education and technology, NATO ASI Series. Series F and Systems Sciences, Vol. 121, 159-186. London: Springer-Verlag.

Fowler, H., & Fowler, F. (1990). *The concise Oxford dictionary of current English.* Oxford: Clarendon Press.

Gallo, C. (2011). *The innovation secrets of STEVE JOBS.* New York: McGraw Hill.

Greenhow, C. (2009). Windows on the future. Learning and Leading with Technology, 36(6), 10-11.

Herman, M., & Laumakis, P. (2009). *Using the CBR to enhance graphical understanding.* Mathematics Teacher, 102(5), 383-389.

Lessem, R., & Schieffer, A. (2010). *Integral research and innovation.* Surrey, England: Gower Publishing Limited.

Mishra, P., & Koehler, M. (2009). *Using the TPACK framework: You can have your hot tools and teach with them, too.* Learning & Leading with Technology, 36(7), 14-18.

Mokros, J., & Tinker, R. (1987). *The impact of microcomputer based laboratory on children's ability to interpret graphs.* Journal of Research in Science Teaching, 24, 369–383.

Nemirovsky, R. (1994). *Slope, steepness and school mathematics.* Proceedings of the 18th PME Conference, Lisbon, Portugal, Vol. III, 344–351.

Obara, S. (2010). *Students' investigation of a view tube.* Mathematics Teacher, 104(2), 127-132.

Pea, R. (1985). *Beyond amplification: Using the computer to reorganize mental functioning.* Special Issue: Computers and education. Educational Psychologist, 20(4), 167-182.

Rogers, Y., & Scaife M. (1997). *Distributed cognition* COGS, University of Sussex, February 1997.

Salomon, G., & Perkins, D. & Globerson, T. (1992). *Partners in cognition: Extending human intelligence with intelligent technologies.* Educational Researcher, 20(3), 2-9.

Shulman, L. (1986). Educational Researcher, 15(2), 4-14.

Thornton, R., & Sokoloff, D. (1990). *Learning motion concepts using real-time, microcomputer based laboratory tools.* American Journal of Physics, 58(9), 858–867.

MATHEMATICA™: PANDORA'S BOX OR CLASSROOM EMPOWERMENT?

Brenton R Groves

Independent Researcher

There are several economic ways for teachers and students to acquire a copy of Mathematica™ for their own use. Learning to program Mathematica™ is identical to becoming fluent in a foreign language; one must learn the vocabulary and grammar. Mathematica™ has an advantage in that each 'word' has one meaning and the grammar is extremely flexible. Wolfram/Alpha produces Mathematica™ syntax from ordinary English requests. This paper contains a number of simple demonstrations of programming, the Modify process, and slide show generation. It will be available on the web so teachers can investigate the material at their own pace.

Methodology

IMPORTANT: Download this PDF format presentation on to your hard disk from the following URL through any search engine

http://dl.dropbox.com/u/49383304/MAV.2012.Programming.pdf

The reason for using the PDF file on your own disk is that the blue URLs can then be downloaded with a Ctrl/Click. This saves a lot of typing.

Mathematica Offers

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Simple Tutorial Videos off The Web – *Mathematica* is not required

The URL for these videos has the format http://www.wolfram.com/broadcast/video.php?channel=86&video=XXX

Each title has the running time followed by its XXX number. For example: Jon McLoone, *Mathematica* Basics (12:15 – 489) can be downloaded from

http://www.wolfram.com/broadcast/video.php?channel=86&video=489

Cliff Hastings's 8-Part screencast series Getting Started with *Mathematica*: runs from 862 to 869.

- Part 1: how to get started using notebooks. (4:12 - 862)
- Part 2: different methods for getting started with *Mathematica*. (10:14 - 863)
- Part 3: getting started with basic calculations.(7:58 - 864)
- Part 4: getting started with basic graphics. (9:45 - 865)
- Part 5: how to make interactive graphics and models (5:03 - 866)
- Part 6: how to utilize data. (3:48 - 867)
- Part 7: how to create presentations (3:38 - 868)
- Part 8:building an example presentation complete with calculations, graphics, and data (4:02 - 869)

All videos can have a soundtrack in Chinese, Japanese and Spanish as well as in English.

The Virtual Book

The Virtual Book is a browsable electronic collection of all he *Mathematica* tutorials, grouped according to functionality.

If they were made into a book, it would be over 11,000 pages long! 'How to' use the Virtual Book and a short video can be found at:

http://reference.wolfram.com/mathematica/howto/UseTheVirtualBook.html Direct link to the video:

http://www.wolfram.com/broadcast/screencasts/howtousethevirtualbook/?w=816 &h=588 (1:12 minutes)

Beginner's Self-Paced Study Course

The beginner's self-paced study course can be found at:
http://www.wolfram.com/training/courses/edu001.html

Learn how to improve your classroom experience with *Mathematica*. This course gives a tour of functionality relevant to teaching and learning, along with case studies and best-practice suggestions for course integration. Topics include making your classroom dynamic with interactive models and a survey of computation and visualization capabilities useful for teaching practically any subject at any level. Level: Beginner This course is available on demand (33:44 minutes). It is free.

Syllabus for Self-Study

- Download the Tutorial References notebook at http://dl.dropbox.com/u/49383304/TutorialReferences.nb
- Save TutorialReferences.nb to your hard disk.
- Click on TutorialReferences.nb to open the notebook in *Mathematica* 8.
- Click on the Help pallett and then on Virtual Book. The screen should look like Figure 1.

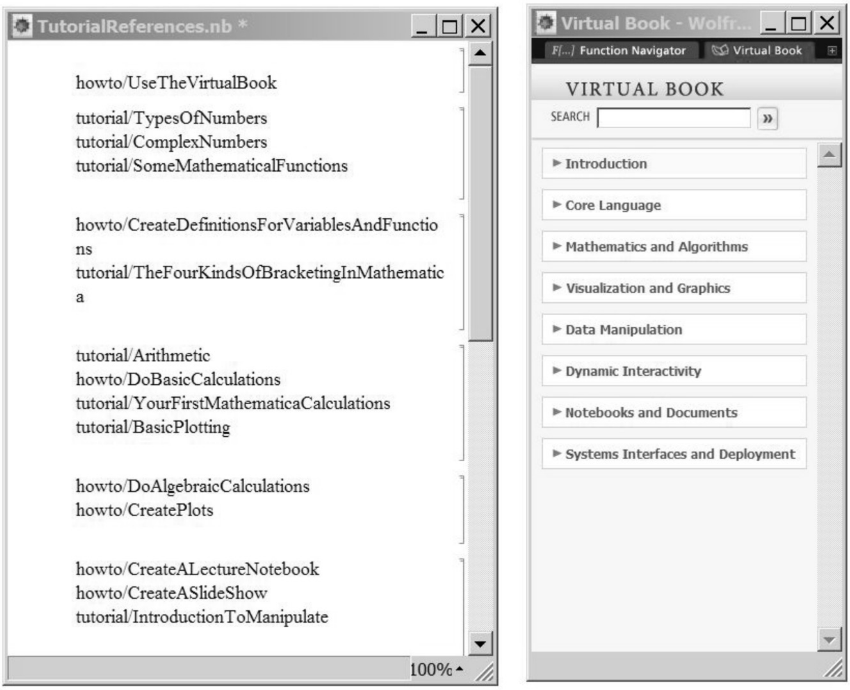

Figure 1. Self-study and Virtual Book Contents

Paste each title in to the Virtual Book search box. These notebooks can be converted into interactive lessons by clicking on 'Delete All Output' in the 'Cell' palette. All Virtual Book notebooks are available as web pages if you do not have *Mathematica*, but they are not interactive.

Advanced Programming

How to Create a Lecture Notebook

Mathematica's slide shows are ideal for use in the classroom, and can very quickly be leveraged as a lesson or lecture. Any presentation created with *Mathematica* can display live interactive content that you can alter, and even create, while presenting. This lets your classes be truly dynamic and provides an unparalleled opportunity to involve your students in the material.

How to Create a Slide Show

You can create and present slide shows directly from within *Mathematica*. *Mathematica* provides an integrated workflow from initial experimentation to final presentation. *Mathematica*-based presentations can contain interactive interfaces and live computations, letting your audience see the effects of changes to parameters in real time.

Tutorial/Introduction To Manipulate

The single command Manipulate lets you create an astonishing range of interactive applications with just a few lines of input. Manipulate is designed to be used by anyone who is comfortable using basic commands such as Table and Plot: it does not require learning any complicated new concepts, nor any understanding of user interface programming ideas.

The output you get from evaluating a Manipulate command is an interactive object containing one or more controls (sliders, etc.) that you can use to vary the value of one or more parameters. The output is very much like a small applet or widget: it is not just a static result, it is a running program you can interact with.

This tutorial is designed for people who are familiar with the basics of using the *Mathematica* language, including how to use functions, the various kinds of brackets and braces, and how to make simple plots. Some of the examples will use more advanced functions, but it is not necessary to understand exactly how these work in order to get the point of the example.

Comments

A PowerPoint slide show of this paper with images and teaching notes, which was presented at the MAV 2012 Annual Conference, can be downloaded from:
http://dl.dropbox.com/u/49383304/MAV.2012.Programming.PPS
Comments and criticisms on this paper are welcome and will be acknowledged. Email: grovesbr@optus.net.au

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Additional References

Wolfram *Mathematica* Tutorial Collection http://www.wolfram.com/learningcenter/tutorialcollection/

Mathematics and Algorithms, 461 pages.

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These tutorials can be downloaded in PDF format or purchased in hard copy for USD$20 each.

Mathematica Books

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MATHEMATICS INVESTIGATIONS IN THE PRIMARY CLASSROOM

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National Institute of Education, Singapore

Mathematical problem solving is at the centre of the framework of the mathematics curriculum in Singapore, and mathematical problems involve non-routine, open-ended and real-world problems. In Singapore, primary mathematics teachers tend to focus on the teaching of various problem-solving heuristics than to the more extended mathematical processes in investigations. In fact, many primary school mathematics teachers are unfamiliar with the investigative processes in mathematics, and have seldom attempted to integrate mathematics investigation into their teaching. This article shares some approaches in the use of mathematics investigations in a primary classroom.

Introduction

Mathematical problem solving is at the centre of the framework for the mathematics curriculum in Singapore. The framework was constructed in 1990 and has not changed much over the years. Besides aiming to equip all primary school students with the mathematical concepts and skills, and develop positive attitudes towards mathematics, the syllabus also aims to help all students develop their process and metacognitive skills so that they can formulate and solve problems. (MOE, syllabus 2007). In fact, mathematical problem solving is the ultimate aim of mathematics education in Singapore. Unfortunately, in most primary mathematics classrooms in Singapore, mathematical problem solving has been limited to solving word problems and non-routine problems. The teaching of problem solving is basically the teaching of the problem solving heuristics and mathematics investigation is unheard of for many teachers.

In the previous syllabus document, a problem covers a wide range of situations from routine mathematical problems to problems in unfamiliar contexts and open-ended investigations that make use of the relevant mathematics and thinking processes (MOE, 2001, p.10). Since then, the term 'open-ended problem' has displaced 'open-ended investigations' in the 2007 syllabus documents. In Singapore, most teachers consider an open-ended problem as a process problem with a unique goal that can be reached by application of some problem solving heuristics. Consequently, teaching of problem solving in primary schools involves the teaching of problem solving heuristics listed in the syllabus. The heuristics include using a representation (e.g., drawing a diagram, systematic listing), making a guess, walking through the process (e.g., act it out, work backwards) and changing the problem (e.g., consider a simpler case).

Problem Versus Investigation

How is a problem different from an investigation? In an open-ended problem, students have limited freedom to decide the goals they want to achieve. For example, the following task has a unique goal (Figure 1). Pupils need to use heuristics (systematic listing) to reach the goal.

How many squares can you find in the following figure?

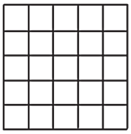


Figure 1. How many squares problem.

This is an example of a process problem commonly given to the students. To challenge the more able students, teachers may pose extensions to the problem by asking 'What if we have the chess board instead of the given 5 units by 5 units square grid?' 'What if a unit square is removed from each corner?' or 'What if the figure is made of triangles instead of squares?' In each question, the goal is given. That is, finding the number of squares or triangles. Students may use different heuristics to solve the problem but all will arrive at the same correct answer. Compare to the following task.

Investigate the squares on the chessboard.

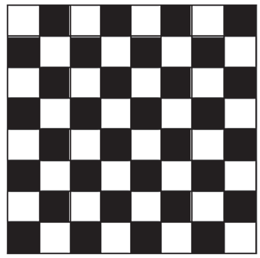


Figure 2. Investigation squares on the chessboard.

This is an investigation (Figure 2). There is no specific and recognizable goal in the task statement. It is up to the students to explore and determine the goals for themselves. Some students may want to examine the squares formed, or squares formed on the chessboard without the two white squares at the corners. Others may want to explore how the black squares are enclosed by rectangles on the chessboard. For example, as shown in Figure 3, eight black squares can be enclosed by an 8 by 2, 5 by 3 rectangles or a 4 by 4 square.

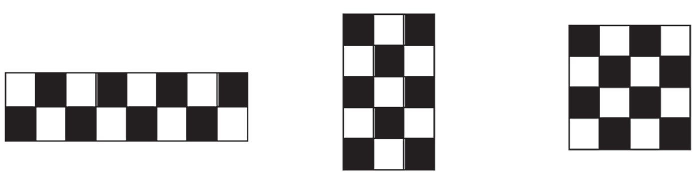


Figure 3. Examples of rectangles.

In USA, the term 'open problem' is often used to describe such task that has no specified goal in the problem statement. It is up to the students who respond to the investigation to decide and formulate a goal. Hence an open problem is an investigation. It is the specification of the goal that distinguishes between an investigation and an open-ended problem as pointed out by Orton and Frobisher (2004). In this article, we view an investigation as an open problem and report the performance of some 'returning' primary school teachers in a mathematics investigation. 'Returning' primary school teachers are teaching-diploma holders who take a break from teaching and return to the National Institute of Education, Singapore, to pursue a degree.

Teachers' Views of Mathematics Investigations in Singapore Primary Schools

The responses found in the portfolios of these teachers submitted at the end of the course seem to indicate that teachers in Singapore seldom implement mathematics investigations in their school mathematics curriculum. Investigation is often ignored or relegated to the category of 'if-have-time activities'. One of the reasons given is the lack of curriculum time for such activity.

> *As the idea of investigation task is crucial to both the study of mathematics and the extension of knowledge in various fields, students in primary schools are not exposed to these as teachers in school are hard-pressed for time to complete the syllabus. (Teacher G)*

There are also teachers who had not done any mathematics investigations in their school days and had the preconceived idea that investigation is meant for science and not mathematics.

> *I have to say that this is one of the most foreign components in this module. I have dealt with plenty of scientific investigations, but have little contact with mathematical investigation. (Teacher J)*

> *I have always thought investigation can be done on Science topics and it is not feasible to get pupils to do on math topics but this lesson has proved me wrong. (Teacher N)*

Some teachers also find investigations tedious as they have to explore various cases in order to make conjecture and 'proof' the conjecture. Very often, they are not confident whether the path adopted and the generalization deduced was valid or not. They find the investigation process tedious and time-consuming. The uncertainty and lack of quick solution can lead to frustration and loss of interest.

> *The session on mathematical investigations was the most challenging for me since it is an open-ended process that might be tedious at times. Furthermore, I don't recall myself doing such investigative tasks during my school days when I learnt mathematics so it is something rather novel to me. (Teacher E)*

Despite the difficulties faced in carrying out the investigation, the teachers do consider mathematics investigation as important and a means to teaching the process skills. They are aware that mathematics investigations, like problem solving, are mathematical activities that exemplify a teaching approach that promotes mathematical thinking, reasoning and communication, and process skills delineated in the curriculum framework. Mathematics investigations are relatively more challenging than problem solving, and often involve different strands of mathematics, and may vary both in style and context. For example, investigation of the symmetry of polygons links geometry to whole numbers. Some investigations may relate to applications in everyday life situations while others may not. Some can be presented to the students as an open problem to be solved, while others can be posed as a question to be answered or an issue to be explored.

Processes in Investigations

The process of problem solving involves Polya's four steps: Understanding, Plan, Do and Check. It involves the application of problem solving heuristics to solve a given problem. Polya's steps to problem solving are repeatedly illustrated in the textbooks while the investigative processes are not found anywhere in the textbook series. Neither is the investigative process stated explicitly in the Singapore mathematics framework. Orton and Frobisher (2004) identify four general processes in mathematics investigations (operational processes, recording processes, reasoning processes and communication processes) that contribute to the mathematical processes involved in the development of new ideas and the exploration of relationships. These mathematics processes are unique to mathematics and include the processes of guessing, pattern searching, making prediction and conjecture, testing conjecture and hypothesizing, testing hypotheses and ending with generalizing and proving (Orton & Frobisher, 2004). Often the processes of making and testing conjecture need to be carried out repeatedly before generalization can be made. Unfortunately, many teachers themselves find these processes tedious, reflecting their beliefs about the nature of mathematics and the teaching and learning of mathematics.

The process of investigation begins with coming to grips with the task and exploring the different problems or aspects inherent to the task so that one or more courses of approach may emerge. Guessing is the result of this exploration. One then investigates systematically by collecting relevant data, tabulate the data so that one can search for pattern(s) among data, predict and make conjecture base on the pattern(s) detected. Before one can make generalization base on the data, the conjecture has to be tested, modified and tested again. Primary school students are not expected to provide a formal proof for their generalization but they should be encouraged to provide some form of justification, for example, testing their generalization by drawing.

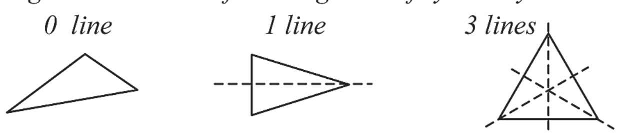
Teaching Mathematics Investigations

If the teachers themselves are not aware of the processes in mathematics investigations and have not carried out any mathematics investigations before, then getting them to incorporate investigations in their mathematics curriculum would be a great leap for them. They would be reluctant to take the leap. One way to remedy this is to remodel the mathematics education courses for the pre-service teachers to involve investigations, and the other way is to provide both moral and material support for those practicing teachers who are keen to take the first step.

At the initial stage, students would not like to carry out investigations as they have been conditioned to identify the 'given' and the 'to find' in a task and then following certain procedure to the answer which is to be found at the back of their textbooks. They are often frustrated because they do not know where to start and the solution cannot be found in a short time. For these students, teachers may want to start with a structured activity sheet to familiarize students with the investigation processes. For example, instead of giving the investigation in Figure 4, Grade 4 students may be guided to investigate the lines of symmetry for quadrilaterals, pentagons and hexagons where each investigation process is highlighted.

<u>Polygon Symmetry</u>

Triangles can have the following lines of symmetry:



1. *Can a triangle have 2 lines of symmetry?*
2. *How many lines of symmetry can a quadrilateral have?*
3. *How many lines of symmetry can a pentagon have?*

Investigate the number of lines of symmetry that are possible for other polygons.

Figure 4. Polygon symmetry (adapted from Sharp & Wilson, 1987).

In the structured activity sheet, prompts are provided to guide students to:

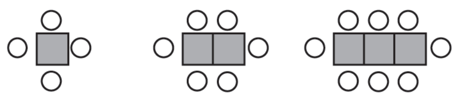
- investigate systematically, starting with triangles and quadrilaterals
- collect more data by considering other polygons such as pentagons, hexagons and heptagons

- tabulate the data collected, look for pattern(s) and make hypotheses
- test their hypotheses on other polygons such as nonagons
- provide summary of their investigation

Once the students are familiar with the investigation processes, short investigations such as Table Seating (Figure 5) can be assigned.

<u>Table Seating</u>

In a restaurant, guests can be seated at square table like these.



Investigate the table seating.

Some questions you may ask:

- *How many people can be seated at 10 square tables?*
- *How many square tables are needed to seat 26 people?*
- *Report your results systematically.*
- *Explain how you could predict the number of people that can be seated for any number of tables?*

Figure 5. Table seating – An investigation.

The task above was assigned to the returning teachers who were not well versed in investigations. The suggested questions caused those who were not risk takers to follow the directions closely and to end their investigation once they had answered the questions. Figure 6 shows sample of such responses given by these teachers. For these teachers, the objective of the investigation was to get a 'neat' solution. They did not consider the practicality of having just a long row of table in a restaurant and they did not explore further. Further exploration would involve more data to be collected and analysed which may not lead to a satisfactory conclusion. They were satisfied once they could derive a generalization for any given number of tables. These responses show lack of critical thinking. The mathematical solution, albeit correct, is not interpreted in the context of the real-world situation, it cannot not be applied to the real-world situations.

Step 3: Tabulate the data

Number of tables	Number of people
1	4
2	6
3	8
4	10
5	12
6	14

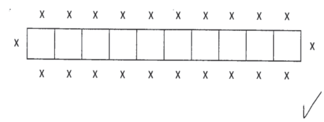
From the above, my hypothesis is:

The number of people seated at 'x' square tables will be 2x + 2.

Step 5: Test my hypothesis

According to my hypothesis, 22 people can be seated at 10 square tables.

$2(10) + 2 = 22$



My prediction is therefore correct.

Figure 6. Table seating – Sample of response I.

However, the lack of specifics in the task did allow some teachers to consider other possibilities leading to more than one conclusion. Some of them explored different arrangements of the tables, while others considered different combinations as illustrated in Figure 7.

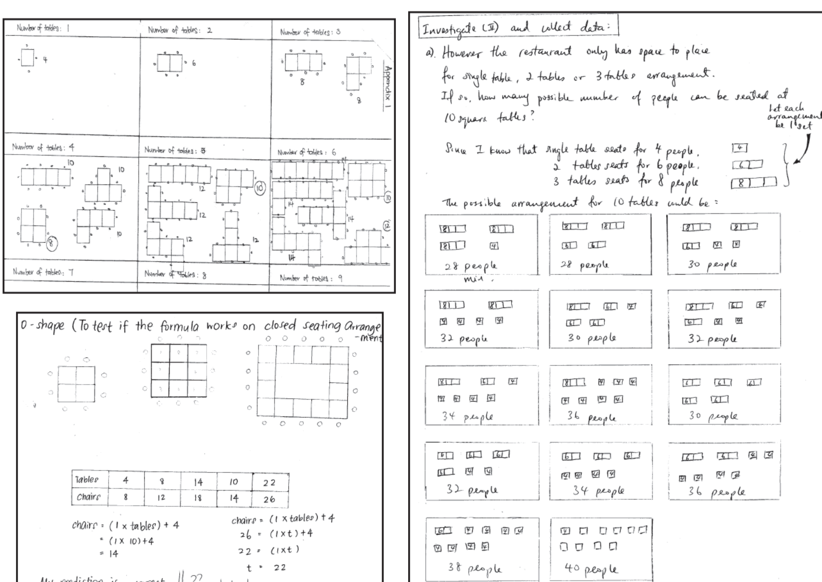
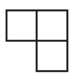


Figure 7. Table seating – Sample of response II.

Often these teachers were not able to test their predictions and make generalization in some cases. Their reports reveal their relatively better organization and communication skills, ability to pose problems and their willingness to take risk. Over the time, after experiencing some success in short investigations, students like these teachers, will be more willing to tackle longer investigations. When given tasks like investigating the figurate numbers or pentominoes (Figure 7), they would not be asking their teachers, "What am I supposed to do?" "What is the question?" "How do we start?" They will not be looking for short-term rewards and are willing to face impasses.

An Investigation

Pentominoes are made up of five identical squares joined edge to edge like this  *and not like these.* 

Investigate pentominoes.

Figure 8. Mathematics Investigation Task

Using Mathematics Investigations in the Classroom

There are many books available where teachers can find examples of mathematics investigations (e.g. Kirkby, 1987; Bastow, Hughes, Kissane, & Mortlock, 1984). Teachers can also design investigation tasks from other subject areas including arts and crafts. For example, origami is good for investigation in geometry and measurement. Many factors have to be taken into consideration when selecting an investigation suitable for students. They include the objective of giving the investigation, the experience, mathematical knowledge and attitudes of the students, and the time constraints. Moreover the format of the investigation must be interesting and relevant resources materials must be made accessible to students. For example, in the investigation on Polygon Symmetry, template of various polygons would help students' exploration while in the investigation on Table Seating, square tiles, cm-cubes and dot grid paper would be useful to students in their exploration.

In the classroom, students can take unpredictable approach in an investigation task. Hence, a teacher must be prepared by having a good understanding of the investigation and its possible solutions. For students who are familiar with the investigative processes, focus questions which could limit the thought processes of the students should be avoided. To emphasize variety and creativity and to promote critical thinking, directions should be kept general to provide students opportunity to make choices and explore. However, such general directions may not be popular with students who are result-orientated and do not tolerate the ambiguity in the task. They are frustrated with the unspecified goal. They do not like making choices as it is difficult for them to decide what they are supposed to do. It would take some time to change their attitudes. Parents too may not accept mathematics investigation as a form of learning. If it forms part of assessment, they question both the validity and objectivity of such assessment tasks, and if it is not part of the assessment, they consider it as a waste of both the curriculum time and their child's effort since the mathematical content of investigation are often not clear to them. The Ministry of Education and the school have to work collaboratively to 'educate' these parents, help them to understand the purpose of investigations.

Conclusions

For a start, mathematics investigations can be part of learning. They do not have to be long. In the primary mathematics classrooms, short investigations could be suitable alternatives to homework worksheets. They are not mundane drill and practice exercises and give students freedom to choose their path of investigation and provide opportunity for them to show what they can do, practice a range of process skills and learn to work effectively with others. Investigation skills do not develop overnight. Teachers need to explicitly teach the skills to the students and help them refine the skills. Early in their experience with investigations, students may need help to identify some problems for them to consider. Such scaffolding can then gradually be removed.

Investigations should not be fillers or activities for only the high achievers. All students irrespective of their mathematical ability should have opportunity to carry out investigations. Investigations can be a means for the students to develop 21st century skills (Ministry of Education, 2010). They enable students to think independently and critically, take calculated risks, persevere, and strive for excellence. Teachers are strongly encouraged to carefully plan and integrate investigations into their school mathematics curriculum, to develop process skills and promote greater diversity and creativity in learning.

References

Bastow, B., Hughes, J., Kissane, B., & Mortlock, R. (1984). *40 Mathematical investigations.* Nedlands, W.A.: The Mathematical Association of Western Australia.

Kirkby, D. (1987). *Maths investigations. Teacher's guide.* London: Hutchinson Education.

Ministry of Education Singapore. (2010). *Nurturing our young for the future: Competencies for the 21st century.* Singapore.

Ministry of Education Singapore. (2007). *Mathematics syllabus (Primary).* Singapore: Curriculum Planning and Development Division.

Orton, A., & Frobisher, L. (2004). *Insights into teaching mathematics.* London: Continuum International Publishing.

Sharp, K., & Wilson, I. (1987). *Coursework mathematics, investigations and problem solving activities.* London: Unwin Hyman Ltd.

MATHEMATICS COMPETITION QUESTIONS AND PROBLEM SOLVING EXPERIENCE

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Educators generally associate mathematics competition with nurturing the mathematically gifted. This chapter demonstrates that mathematics competition questions can provide students with rich learning experience associated with mathematical problem solving, which is the heart of the mathematics curriculum in many countries. The examples used for illustration in this chapter are taken from the Singapore Mathematical Olympiad and the Australian Mathematics Competition.

Introduction

The key role of mathematics competitions in developing the mathematically gifted and the high ability students has been identified by many mathematics educators and mathematicians (for examples, Bicknell, 2008; Campbell & Walberg, 2010; Kalman, 2002). Xu (2010) describes that mathematics competition can be used to "improve the mathematical thinking and technical ability in solving mathematical problems" (p. v) for the higher ability students. Many mathematicians share the same opinion as Xu (2010).

Since the start of the International Mathematical Olympiad held in 1959, mathematics competitions at the national levels have been used in identifying and supporting mathematical talents for a country, building the national pool of mathematically gifted students and preparing the best among them for the International Mathematical Olympiad, the most prestigious mathematics competition internationally.

Recently, some mathematics educators are beginning to recognize the usefulness of mathematics competition questions for developing students' (not necessarily restricted to the mathematically talented students) learning of mathematics (for example, Toh, 2012; Vistro-Yu, 2008). Toh (2012) has identified mathematics competition questions potentially could be used to arouse students' interest in mathematics and to engage them in developing mathematical reasoning and higher order thinking skills. Vistro-Yu (2010) has advocated the generating of "new" mathematics problems by innovating on existing mathematical problems in her teacher professional development workshops. She tapped on the rich resource of mathematics competition questions from the past years in her discussion.

In this paper, it will be shown how mathematics competition questions can be used to enrich students' learning of mathematics through equipping them with the learning experience associated with mathematical problem solving.

Mathematics Competition and Problem Solving

The author's personal association with mathematically talented secondary school students and trainers of mathematics competitions shows that the vast collection of the past year mathematics competition questions are used solely in preparing the mathematically talented students for mathematics competitions by equipping them with additional content knowledge or "resources" (Schoenfeld, 1985) not usually covered in the mainstream school mathematics curriculum. In other words, these competition questions serve as a guide for the potential contestants and trainers on the additional content knowledge that students participating in the mathematics competition must be equipped (as there is no official "syllabus" for the competitions).

Teachers generally do not tap into the mathematics competition questions for their teaching. It is believed that the vast collection of mathematics competition questions can benefit only the mathematically talented but not the general student population, hence mathematics competitions have very weak link with the mainstream school curriculum. With this belief, the pedagogical value of the mathematics competition for the general student population is neglected.

In this paper, the author provides an alternative way of looking at this rich resource of competition questions: most of the questions, when used appropriately, have very strong link with the usual mathematics curriculum through mathematical problem solving, which is undeniably the heart of the mathematics curriculum in very countries in the world. For example, in the United Kingdom, the Cockcroft Report (1982) stated that "mathematics teaching at all levels should include opportunities for problem solving". In the United States, the National Council of Teachers of Mathematics (NCTM) in their document stating the principles and standards for school mathematics stated that: "[p]roblem solving should be the central focus of the mathematics curriculum" (NCTM, 1989). In Australia, the 1990 "National Statement on Mathematics for Australian Schools" stated that students should develop their capacity to use mathematics in solving problems individually and collaboratively (Australian Education Council, 1990). Mathematical problem solving is at the heart of the Singapore primary and secondary mathematics curriculum (Singapore Ministry of Education, 2006).

From a review of the past year mathematics competition questions from Singapore and Australia, it is apparent that many competition questions provide a good avenue for equipping students with the processes of problem solving, and helping them to acquire a problem solving model.

Any student attempting mathematical problem solving requires a model to which he or she can refer, especially when progress is not satisfactory (Toh, Quek & Tay, 2008). Even a good problem solver may find the structured approach of a model useful, as Alan Schoenfeld (1985) recounted in the preface to his book *Mathematical Problem Solving* about Polya's book *How to Solve It*:

In the fall of 1974 I ran across George Polya's little volume, *How to Solve It*. I was a practising mathematician ... My first reaction to the book was sheer pleasure. If, after all, I had discovered for myself the problem-solving strategies described by an eminent mathematician, then I must be an honest-to-goodness mathematician myself! After a while, however, the pleasure gave way to annoyance. These kinds of strategies had not been mentioned at any time during my academic career. Why wasn't I given the book when I was a freshman, to save me the trouble of discovering the strategies on my own?

In this paper, Polya's problem solving model will be used as the base of discussion for several reasons: Polya's model is well-known and it is the model described in the syllabus document of the Singapore Ministry of Education (which the author is most familiar) and many other countries. Further, the model is relatively easy for students to 'carry about' in their heads. Of course, any other sound problem solving model is equally viable for discussion.

Polya's model could best be depicted as a flowchart with four components, *Understand the Problem, Devise a Plan, Carry out the Plan*, and *Check and Extend* (the original words used by Polya are *Looking Backward*). To reflect the dynamic nature of mathematical problem solving, the four stages of the model should not be seen as a linear sequential one; back-flow between any two of the four stages should be allowed.

Acquiring Problem Solving Via Competition Questions: Two Examples

We present two questions from the past years' mathematics competitions: one from the Australian Mathematics Competition (AMC) and one from the Singapore Mathematical Olympiad (SMO), and demonstrate how engaging students to solve these questions could provide students with rich learning experience of the mathematical problem solving process.

Question 1. *While attempting to solve a quadratic equation, Christobel inadvertently interchanged the coefficient of x^2 with the constant term, causing the equation to change. She solved this different equation accurately. One of the roots she got was 2 and the other was a root of the original equation. Find the sum of the squares of the two roots of the original equation.* (AMC)

Stage 1: Understand the Problem

This is a "non-routine" problem; secondary school students have probably not encountered this genre of questions in the mainstream school mathematics. Problem solving necessarily begins with one trying to understand the problem (which is the first stage of Polya's model). In the context of this question, what is meant by "interchang[ing] the coefficient of x^2 with the constant term"? One crucial strategies in trying to understand

E.g. $3x^2 + 2x + 4 \rightarrow 4x^2 + 2x + 3$

$5x^2 + 4x + 1 \rightarrow x^2 + 4x + 5$

$6x^2 + 5x - 1 \rightarrow -x^2 + 5x + 6$

By considering specific examples as illustrated above, one is likely to have a better understanding of the procedure described in the problem and proceed to the next stage of Polya (of devising a plan).

Stage 2: Devise a Plan

To solve this problem, the following strategies are required:

- *Setting a sub-goal*: It is easily observed that the key to unlock this problem would be to study the effect on the solution of a quadratic equation when the coefficient of x^2 and the constant term are interchanged
- *Consider specific numerical examples*: One might consider specific quadratic equations whose roots are easy to determine to examine the effect of interchanging the two coefficients.

The above "strategies" are what we call *heuristics* in the language of problem solving. The use of *heuristics* comes in handy at this stage of devising a plan. According to Schoenfeld (1985), a heuristic is a 'rule of thumb' for making progress in difficult situations when one solves a mathematical problem. The teaching of heuristics is an important (but not the sole) aspect of mathematical problem solving. The right choice of heuristics is a necessary facet of successful problem solving. Not only must one be acquainted with a problem solving model and a list of heuristics, a good problem solver must be able to manage resources at his or her disposal, choose promising heuristics to try, control the problem solving process and progress, and examine his or her beliefs about mathematics that hinder or facilitate problem solving (Schoenfeld, 1985).

Stage 3: Carry out the Plan

For this question, it is a natural progression to proceed to Stage 3 by considering specific quadratic equations whose roots are easy to find. Since we want to examine the effect of switching the coefficient of x^2 and the constant term, one's choice of the quadratic equation should have these two terms to be distinct. Consider the following examples:

$9x^2 + 6x + 1 = 0$ (roots are -3, -3)	$\rightarrow$	$x^2 + 6x + 9 = 0$ (roots are $-\frac{1}{3}, -\frac{1}{3}$)
$x^2 - 5x + 6 = 0$ (roots are 2, 3)	$\rightarrow$	$6x^2 - 5x + 1 = 0$ (roots are $\frac{1}{2}, \frac{1}{3}$)
$4x^2 - 5x + 1 = 0$ (roots are 1, $\frac{1}{4}$)	$\rightarrow$	$x^2 - 5x + 4 = 0$ (roots are 1, 4)

From the above examples one would *conjecture* that the roots of the new equation obtained by switching the coefficients of x^2 and the constant term are the reciprocals of the original equation. To further convince oneself, one could consider several other numerical examples

of quadratic equations to *verify* this observation. Making *conjectures* and *verifying conjectures* are important heuristics of problem solving.
Since "*[o]ne of the roots she got was 2*", if our conjecture was correct, the corresponding root of the original equation was $\frac{1}{2}$ and that since "*the other was a root of the original equation*", this original root must be invariant under reciprocal. Thus, the choice of this original root must be 1 or −1. There is insufficient information in the question to determine which must be the original root of the equation.

"*Find the sum of the squares of the two roots of the original equation.*" The possible original roots of the equation are either 1 and $\frac{1}{2}$; or −1 and $\frac{1}{2}$ In either of the two cases, the sum of squares of the roots is $\frac{5}{4}$.

Stage 4: Check and Expand

After obtaining the answer to the original problem, student should be encouraged to reflect on their method and the answer to the problem (this reflection part is usually absent in competition training for students). For instance, the heuristic of considering specific quadratic equations and then recognising the pattern to obtain the answer to the given problem (as shown in Stage 3 above) is not rigorous, as pattern recognition may fail (see, for example, Toh, Quek, Leong, Dindyal & Tay, 2011, pp. 125, which is a classic example). Teachers could challenge their students to appreciate the fallibility of pure pattern recognition and consider alternative solution for this question.

Alternative method: It is crucial that in algebraic reasoning, students must be able to move beyond considering specific numerical examples to a general abstract case (Lee, 2006). Compare the general quadratic equations $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$, where the constant term and the coefficient of x^2 are interchanged. Since the second quadratic equation can also be expressed as

$$a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c = 0,$$

the roots of the first equation is the reciprocal of the second. The advantage of using this rigorous approach is that the earlier approach of considering specific numerical examples can be dispensed with.

Expansion of the problem: Students can be challenged to move on to expand on the given problem; a natural progression for students could be to explore whether such an interchange of the coefficient of the x3 term and the constant term in a cubic polynomial equation would necessarily result in the roots of the new equation being the reciprocal of the first one.

Consider a specific cubic equation, for example, $x^3 + 6x^2 + 12x + 8 = 0$, whose roots are -2, -2, -2. By interchanging the coefficient of x3 and the constant term of the original equation, we obtain

$$8x^3 + 6x^2 + 12x + 1 = 0$$

whose roots are not the reciprocal of the roots of the original equation.

However, if in addition the coefficients of x2 and x are interchanged, the new equation obtained

$$8x^3 + 12x^2 + 6x + 1 = 0$$

has roots $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$, which are the reciprocals of the roots of the original equation. It is a good exercise to challenge the students to offer a rigorous solution for this new problem as in the original question.

Students could further be challenged to explore how the coefficients of a higher degree polynomial equation be interchanged in order that the roots of the new equation are the reciprocal of the original one.

Note that since the main business of mathematics competition is for students to obtain the correct <u>answer</u> in the shortest possible time, such checking and expanding to enhance students' learning or deepen their understanding and appreciation of mathematics could be ignored. Further, strategies to obtain the correct answer might lead students to be contented with partial reasoning.

Question 2. Let x be a real number and let $A = \frac{-1+3x}{1+x} - \frac{\sqrt{|x|-2} + \sqrt{2-|x|}}{|2-x|}$.

If A is an integer, find the unit digit of A2003. (SMO 2003)

Stage 1: Understand the Problem

This is likely to be another "non-routine" problem for a secondary school student. Students need to grapple with the two facts: (1) the given *A* appears to be a function on the real numbers instead of a whole number; (2) for the question to be valid (finding the *unit digit*), the expression A must be a whole number! In order to *understand* the problem, a natural heuristic to use is to *substitute some numbers* for the values of *x*. It would appear that the function *A* is not defined for most values of *x* that one might substitute – this is the "catch" of the problem.

Stage 2: Devise a Plan

The first heuristic needed to solve this question is to *set a subgoal* – to determine all the possible values represented by the expression A by substituting all possible values of *x* for which the expression *A* is defined. Once the value(s) of *A* is (are) determined, *pattern recognition* could be used to find the unit digit of A^{2003}.

Stage 3: Carry Out the Plan

By substituting different possible values of x for which A is defined, one would be quick to realize that the function A is not defined for most values of x. The next step is to *identify the attribute* of the function (it is known that square root of a negative number is not defined). Following this, one would observe that the only choice of $x = -2$. With this, $A = 7$. Thus, finding the unit digit of A^{2003} through pattern recognition follows easily.

Stage 4: Check and Expand

This problem involves two parts: (1) conceptual understanding of radical functions and (2) finding the unit digit of large powers of a whole number. It provides a good opportunity for students to dissect this problem into two parts for discussion:

Radical function: The domain of square root function consists of non-negative real numbers only. This could lead students to explore a large category of problems:

E.g 1. *Find the sum of all the values of real x for which the function* $\sqrt{x^2-2}+\sqrt{2-x^2}$ *is real.*

E.g. 2. *Find the set of values of x for which the function* $\sqrt{2x-x^2-1}$

This category of problems deepens students' understanding of the domain of a function, in addition to engaging them in the problem solving processes.

Finding unit digit of large powers of whole numbers: It is not difficult to generate items getting the solvers to find the unit digit (or even the last two digits) of the powers of large numbers.

E.g. 1. *Find the unit digit of the number* 7^{4009}; 17^{4009}; 1979^{4009}.

E.g. 2. Find the last two digits of the number 7^{2012}; 5^{203954}.

E.g. 3. Find the unit digit of the number 7^{7^7}

Although the level of difficulty varies across the three examples above, the standard heuristic is to solve these problems by pattern recognition.

Use of Competition Questions in Classroom

Many mathematics competition questions, as illustrated by the two questions from the Singapore Mathematical Olympiad and the Australian Mathematics Competition above, provide good resources that teachers could use in the usual mathematics classroom to engage students in mathematical problem solving. Teachers could use this opportunity to equip their students with a problem solving model. As a start to conducting problem solving lessons using mathematics competition questions, it would be useful if (1) students are first familiarized with the language of problem solving that can be used throughout the mathematics classes, and (2) teachers could model the problem solving processes (thereby demonstrating Polya's four stages) when they encounter an unseen problem, thereby highlighting the stages involved in handling non-routine problems.

In the usual mathematics classroom, teachers might consider providing scaffolding for their students in handling the problems once they are familiar with the language of problem solving and a model of problem solving (e.g. Polya's model). As an illustration, Toh et al (2011) has provided a generic form of scaffolding in the form of a "mathematics practical worksheet" for problem solving lessons.

Conclusion

This paper provides readers with examples how mathematics competition questions could provide students with rich learning experience in mathematics in mathematical problem solving. It should be noted that

1. Not all competition questions are good for teaching problem solving. There are usually some questions which are technically involved and need "special" techniques.
2. Students must have sufficient "cognitive resources" related to the questions for discussion.

It is crucial for teachers to identify the questions with sound pedagogical principle and use appropriate scaffoldings to optimize the benefit for their students.

References

Australian Education Council. (1990). *A national statement on mathematics for Australian schools.* Australia: Curriculum Corporation for Australian Education Council.

Bicknell, B. (2008). Gifted students and the role of mathematics competitions. *Australian Primary Mathematics Classroom, 13*(4), 16–20.

Campbell, J., & Walberg, H. (2010). Olympiad studies: Competitions provide alternatives to developing talents that serve national interests. *Roeper Review, 33(*1), 8–17.

Cockcroft Report. (1982). *Mathematics counts.* London: Her Majesty's Stationery Office.

Kalma, R. (2002). Challenging gifted students: The math Olympiads. *Understanding Our Gifted, 14*(4), 13-14.

Ministry of Education. (2006). *A guide to teaching and learning of O-Level mathematics 2007*. Singapore: Ministry of Education, Curriculum Planning and Development Division.

National Council of Teachers of Mathematics. (1989). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Toh, T. L. (2012). The roles of mathematics competition in Singapore mathematics education. Regular lecture presented at ICME12, July 2012, Seoul.

Toh, T. L., Quek, K. S., & Tay, E. G. (2008). Mathematical problem solving - A new paradigm. In J. Vincent, J. Dowsey, & R. Pierce (Eds.), *ConnectED maths: Proceedings of the 45th Annual Conference of the Mathematical Association of Victoria* (pp. 356 - 365). Melbourne: MAV.

Toh, T. L., Quek, K. S., Leong, Y. H., Dindyal, J., & Tay, E.G. (2011). *Making mathematics practical: An approach to problem solving*. Singapore: World Scientific.

Vistro-Yu, C. (2009). Using innovative techniques to generate 'new' problems. In B. Kaur, B. H. Yeap, & M. Kapur (Eds.), *Mathematical problem solving: AME Yearbook 2009* (pp. 185 – 207). Singapore: World Scientific Publishing.

Xu, J. G. (2010). Lecture notes on Mathematical Olympiad courses: For junior section volume 1. Singapore: World Scientific.

LOGIC AND MATHEMATICS

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Developing logical thinking in students; why should we and how can we?
This paper discusses the notion that the teaching or fostering of logical thought is an important prerequisite of successful learning in mathematics. An ability to be more logical in thinking appears to be an indicator as to how successful a student will be in mathematical achievement (Nunes et al., 2007). Nunes et al. (2007) claim that there is a place for specifically teaching logical thinking, which will then lead to improved mathematical understandings in all areas of mathematics. Attribute blocks (or logic blocks, as they are often referred to) can be a very useful tool to facilitate the development of logical thought. Some practical activities for the classroom, that utilise attribute blocks, are also shared.

Background

Piaget is the educational theorist that most of us remember from our educational psychology studies (Krause, Bochner, Duchesne & McMaugh, 2010). Piaget's work has had a significant influence on mathematics education, with the contention that through accommodation and assimilation, children construct schema (Jorgensen & Dole, 2011). Schema theory explores the way in which children make connections between situations, allowing them to transfer understandings developed in one context to another. Piaget's stage theory of cognitive development strongly influenced early childhood and primary education in the 1960's and 1970's. In Piaget's schema, Stage 4, the formal operations period (which occurs between the ages of 11 or 12 to adulthood) is the one in which adolescents fully develop reasoning that is "logical, abstract and systematic" (Wadsworth, 2004, p.120). However, prior to Stage 4, there are clearly many elements of logical thought being developed, such as in classification tasks; even very young children show some logical thought processes. In more recent years, there has been some criticism of Piaget's stage view, with its assertion that progression through these stages cannot be accelerated in any way (Bobis, Mulligan & Lowrie, 2009). Another concern has been that this stage view highlights what students cannot do, rather than what they can do (Jorgensen & Dole, 2011).

Despite these issues, Piaget has most definitely made a vital contribution to the understanding of how number concepts and logic are developed in young children. Piaget's work has had a strong influence on constructivist thinking, which recognises that mathematics must make sense to students if they are to retain and learn mathematical concepts. The role of dialogue and argumentation are also considered crucial in the constructivist paradigm (Bobis, Mulligan & Lowrie, 2009). Further to Piaget's work on constructivism, we have the influence of Vygotsky, who saw the role of the expert teacher as being vital (Jorgensen & Dole, 2011). An expert teacher is one who is able to identify where a student is currently at in their understandings, and then, by providing rich discourse, questioning or appropriate learning situations, encourages students to move on from their current course of thought as well as the development of "logico-mathematical knowledge" (Jorgensen & Dole, 2011, p.25). As stated by Van de Walle & Lovin (2006), "Conceptual knowledge of mathematics consists of logical relationships constructed internally and existing in the mind as a part of a network of ideas" (p.6).

Connection between Logical Thinking and Mathematics Ability

Clements and Sarama (2006) discussed the fact that all thinking involves mathematics and asserted that this all comes down to logic. Although logic may seem like a very abstract thought process, researchers see an implicit use of logic in children at a very young age. Nunes et al. (2007) investigated whether children's mathematical understandings are actually based on their ability to reason logically. In the results of a longitudinal study, they asserted that evidence demonstrates that logical abilities are a powerful predictor of mathematical achievement later in the child's schooling. In addition, Nunes and his colleagues trained a group of students in logical reasoning and found that this group made greater progress in mathematics, than a control group that did not receive any training (Nunes et al., 2007). This study specifically sought to establish a causal link between logical reasoning and mathematical learning with the logic training having a pure focus on logical relations rather than on calculation. Nunes et al. (2007) found that the teaching of logical competence was extremely successful and had a "strong and beneficial effect … even after an interval of 13 months" (p.162). These findings were based on standardised achievement tests used in British schools with seven year olds.

The next two sections will explore how attribute blocks have been used previously in schools and early childhood centres to assist with the development of logical thinking and look at the connection with set theory. There will be a focus on the importance of developing an understanding of set theory when looking at problem-solving, particularly in mathematics.

Use of Attribute Blocks in the Past

Attribute blocks (Invicta, 1980) can be seen as one tool which can be utilised to develop logical thinking. In the 1980's and 1990's, attribute blocks were seen as a very useful piece of equipment which was readily available in most early childhood centres and primary schools (at least in the New Zealand setting). Booker, Bond, Sparrow and Swan (2010) discussed that fact that the need for materials is "fundamental in teaching mathematics" (p.15) in order to build conceptual understanding. The latter part of the twentieth century was an era when teachers often had few materials or hands-on resources to utilise with their children, to assist with the development of mathematical ideas. This was particularly challenging for teachers (and students) when endeavouring to foster complex, conceptual understandings. The lack of available equipment was quite surprising, considering that constructive thought and Piaget's ideas were widely acknowledged and incorporated into teacher training programs. The reality was that in many classrooms, traditional methods still abounded, based on rote learning; 'programs' of learning were frequently textbook driven. As a primary school teacher in the 1980's and 1990's, it was a relief to have some concrete materials to use! Furthermore, attribute blocks were extremely useful as a practical way of developing understandings of sets and how set theory could be used to develop a means of understanding logical connections.

Set Theory

"Sorting according to given criteria is one of the most fundamental processes in mathematics" (Haylock, 2010, p.341). Sorting provides us with the basis of working with and flexibly understanding and interpreting data; it also provides an underlying understanding of what counting is all about. Venn diagrams were developed by John Venn

(1834-1923), as a useful aid in representing logical relationships between various sets. The languages of sets (such as universal, intersection and complement) (Haylock, 2010) are all clearly portrayed through Figure 1.

- 'Intersection' is the area where two or more sets overlap e.g. the part where the 'Red' and 'Large' sets overlap, contains Red, Large objects.
- 'Universal' refers to the entire set.
- 'Complement' refers to the empty set i.e. one without any objects in it.

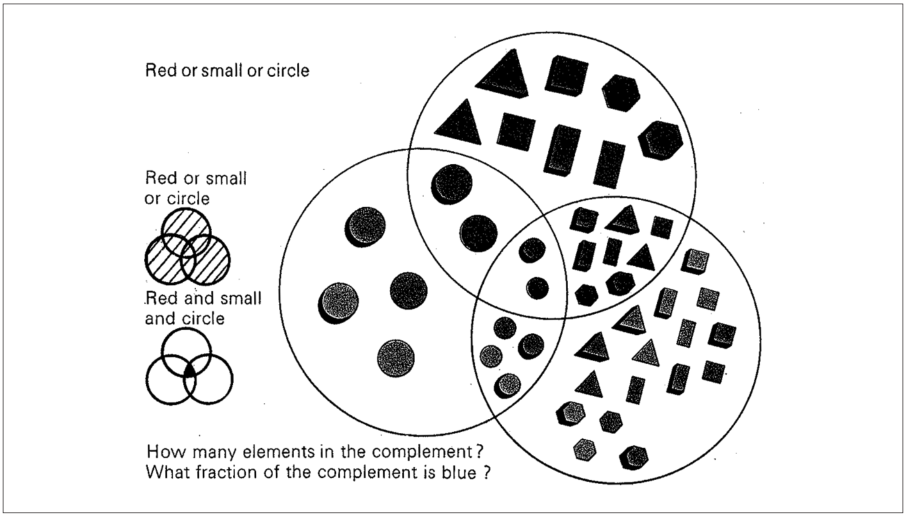


Figure 1. Sorting attribute blocks (Darker shapes represent red).

Another diagrammatic form, which can be used to assist with understanding the relationships between sets, is the Carroll diagram, which was created by Lewis Carroll (1832- 98). Carroll is more widely known as the author of the Alice books but was also a mathematician who took a great interest in the development of logical thought (Haylock, 2010). Carroll devised a diagram which is an example of a two-way reference chart, which allows for sorting of data in response to two variables (see Figure 2).

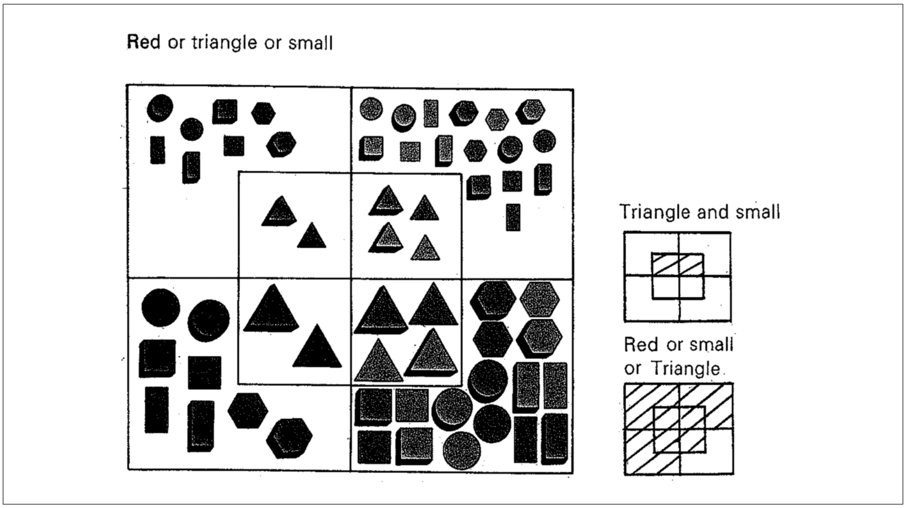


Figure 2. A Carroll diagram with attribute blocks.

Venn and Carroll diagrams can both be used very successfully with attribute blocks; they are a tool which can encourage students to use these diagrams to assist their logical thought processes. Once these ideas have been grasped with attribute blocks, these strategies can be easily utilised in other situations that demand logical problem-solving as students have now been provided with some sort of framework to utilise. Problem-solving is increasingly seen as a vital area to focus on in our teaching, as mathematics has moved from consisting of 'a page of sums' to drawing on students' understandings, and then moving on to more abstract processes. This requires a much greater connection between literacy and mathematics which can be a major stumbling block for some students. This current understanding of what mathematics is all about has a strong focus on solving problems and thinking. I have included in the next section a number of activities that can be used with attribute blocks that will foster logical thinking; they can also provide a range of frameworks for applying this thinking in more general problem-solving situations.

Attribute Block Activities

Listed below are some of the attribute block activities that can be used readily with children to encourage logical thinking.

1. Grouping in sets by attribute- e.g. set can be described as 'Triangle and Thin' or 'Red and Thin'. Can question further e.g. "How many are there in the set that contains Yellow and Thick and Small elements?"

2. Create a pattern, then remove some blocks; other students can work individually or co-operatively to find the missing blocks (great for group-work).
3. Complete a pattern using Thin/Thick elements and/or Large/Small etc.
4. One attribute difference game - continue a 'chain' showing one difference between each block.

   Extension: Continue a chain showing two differences between each block or three or four difference.

   Extension: Continue a chain alternating from one to two differences.
5. Two dimensional domino game (see Figure 3).

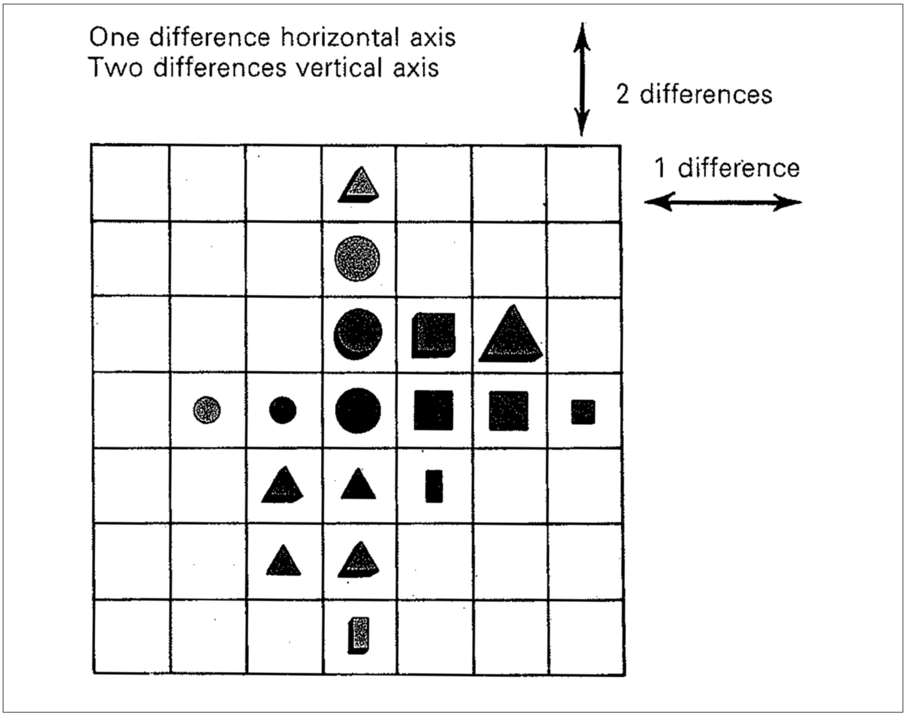
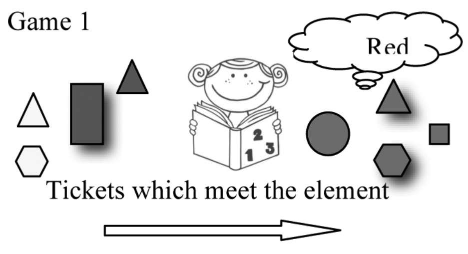


Figure 3. A one and two difference game.

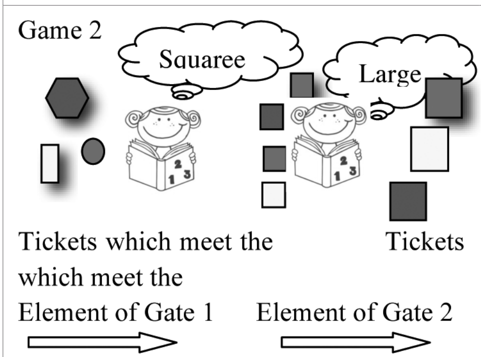
6. Disjoint sets- using grouping circles or sheet divided in two parts. Put all blocks of one element in each set e.g. all red. Describe and/or say how many blocks in the set.
7. Union of two or three sets- Using intersecting sets(see Figure 1).

8. Gate Games (see Figure 4).

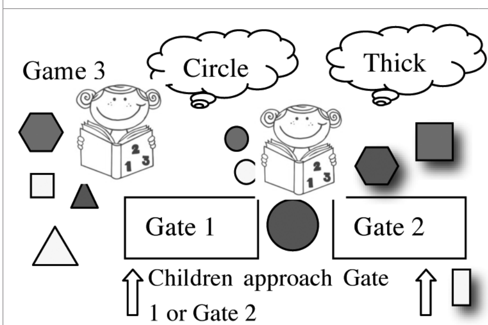
   A gate keeper is chosen by the teacher or children. 'Tickets' are used throughout the games, these are the attribute blocks.



Children choose a 'ticket'. When someone approaches the gate they must show their 'ticket' to the gate-keeper who will let through only those 'tickets' which meet the element he has chosen. Anybody who can correctly identify what is being collected will take the place of the gate-keeper.



The teacher/children pick two gate-keepers for this game. The children choose a 'ticket'. All 'tickets' which meet the first element (e.g. square) chosen by the gate-keeper at Gate 1, will proceed to Gate 2. The second gate-keeper will have chosen a different element (e.g. large). All 'tickets' which meet the second element proceed. The first two people to proceed and identify the 'set' will become the new gate-keepers.



For this variation, the children can take their 'ticket' to either Gate 1 or Gate 2. For example the element of Circle can get past the Gate 1 child and the element of Thick can get past the Gate 2 child.

Figure 4. Gate games using attribute blocks.

9. Using logic arrows - place blocks on chart with arrows indicating the change to make e.g. 2 differences or 3 differences (see Figure 5).

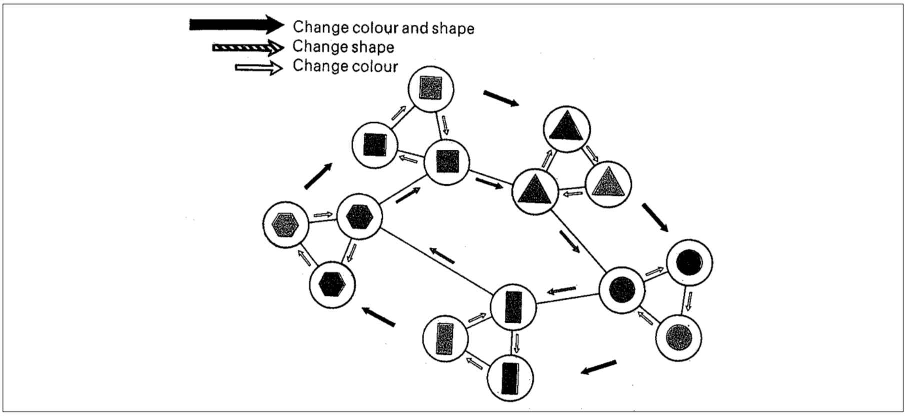


Figure 5. Logic arrows with attribute blocks.

10. Track ways (see Figure 6).

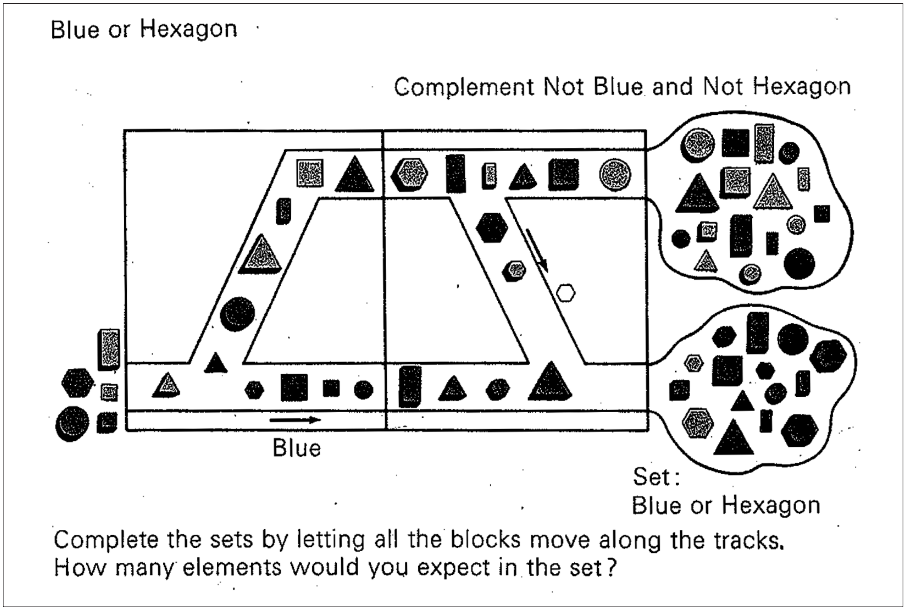


Figure 6. Track ways.

These examples are just a sample of the various activities that can be undertaken with attribute blocks but provide an excellent starting point in developing logical thinking in children. All of these tasks can be attempted individually or in co-operative groups, which would have the additional advantage of fostering "higher academic achievement … positive attitudes … acceptance and understanding of individual differences" (Bobis, Mulligan & Lowrie, 2009, p.317). After establishing the language of logic and exploring these activities with attribute blocks, it would be extremely beneficial to then use the strategies acquired with other problem solving or logic tasks, further developing links with literacy. These strategies can all support young learners "to develop the ability to think creatively, critically, strategically, and logically" (New Zealand curriculum, 2007, p.26).

References

Clements, D., & Sarama, J. (2006).*Your child's mathematical mind.* Scholastic parent and child , 14(2), 30-37.

Bobis, J., Mulligan, J., & Lowrie, T. (2009). *Mathematics for children.* (3rded.), Frenchs Forest, N.S.W: Pearson.

Haylock, D. (2010). *Mathematics explained for primary teachers.* (4thed.) London: Sage.

Invicta. (1980). Early learning and thinking in mathematics. London: Ebenezer Baylis & Son.

Jorgensen, R., & Dole, S. (2011). *Teaching mathematics in primary schools.* (2nded.), N.S.W: Allen & Unwin.

Krause, K., Bochner, S., Duchesne, S., & McMaugh, A. (2010). *Educational psychology for learning and teaching.* (3rded.). Victoria, Australia: Cengage.

Marolda, M. (1994). Exploring attributes. Palo Alto, CA: Dale Seymour.

Ministry of Education. (2007). *The New Zealand curriculum.* Wellington, New Zealand: Learning Media.

Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J. (2007). *The contribution of logical reasoning to the learning of mathematics in primary school.* British Journal of Developmental Psychology, 25, 147-166.

Van de Walle, J., & Lovin, L. (2006). *Teaching student-centered mathematics.* Grades 3-5. Boston, USA: Pearson.

Wadsworth, B. (2004). *Piaget's theory of cognitive and affective development.* (5thed.). Boston, USA: Pearson.

SOLVING REAL-WORLD MATHEMATICS PROBLEMS: AN EXAMPLE FROM SINGAPORE

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The Singapore mathematics curriculum recognizes the importance of engaging students in solving real-world mathematics problems. With the current revision of the school mathematics curriculum, solving real-world problems in the form of mathematical modeling is an essential part of teaching that would be incorporated in the new Singapore school textbooks which will appear in 2013. This paper demonstrates with one example how a real world context in a textbook can be developed into a modeling task, identifies the knowledge required of students in solving the task, and how teachers can guide their students through the processes of scaffolding.

Introduction

Mathematics teachers have the common experience that their students want to know the relevance of classroom mathematics to their lives. Engaging students in solving real-world mathematics problems can thus help to give students the meaning to the mathematical content knowledge that they learn (Williams, 2007). It is thus not surprising that there is a call among the mathematics education community throughout the world to introduce mathematical tasks that are related to 'real life' and the 'real world'.

The new Singapore Secondary School Mathematics curriculum emphasizes the application of mathematics to solve real-world problem. According to the Singapore Ministry of Education, students must be able to connect mathematics that they have learnt to the real world in order to enhance their understanding of key concepts and to develop mathematical competencies (Ministry of Education, 2006a; 2006b).

Real-world Mathematics Problems

Since the early years, Singapore textbooks have been replete with problems on 'real life application', in addition to some context-free questions for student practice. Why then is there a need to highlight this aspect again in the recent curriculum review in 2007?

A survey of the textbook problems on 'real-life application' of mathematics shows that most of these questions present the numerical values of the given data and the final numerical answers in a "very clean and tidy state" (Ang, 2009). It is very difficult to convince students that these are the real life application of mathematics. In addition, real life problems are usually open-ended, as in there may be more than one possible answer to the problem, which contrast the textbook application questions which usually have one final correct answer. Furthermore, in handling many real life problems, the solvers would need to state their assumptions or impose restrictions in order to arrive at a particular answer, while the assumptions in textbook questions have been made ready for the solver.

This leads to the new emphasis of 'mathematical modeling' in the Singapore Mathematics curriculum, which brings out that real life application should involve more than contextualizing a mathematical problem; it should take into consideration that real life tasks are usually without neat answers (and could likely be open-ended), and assumptions need to be made before a particular task can be solved. The answers may be dependent on the assumptions made in the particular context.

In the next section, we demonstrate how a typical textbook context (Figure 1) on the use of proper and improper fractions and mixed numbers which was introduced in a typical textbook could be modified into a modeling task. We shall also demonstrate the processes involved in solving the modeling task, and discuss the pedagogical considerations that teachers need to note.

From Real World Context to Modeling Task: An Example

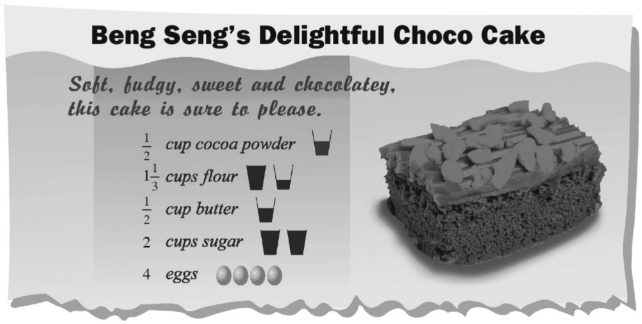

Figure 1. A context from a secondary school textbook (Toh, Lim, Chua & Heng, 2006) on fractions and whole numbers.

In the Singapore context, modelling is usually understood as the process of formulating and improving a mathematical model to represent and solve real-world problems. A 'real-world' problem that uses the context of Figure 1 can be developed as follows:

> Modelling Task
>
> Beng Seng opens a bakery shop in Orchard Road. He has a secret recipe shown above for his famous chocolate cake. Advise him how much he should fix the selling price of a Beng Seng's Delightful Choco Cake.

As in any real-world problem, there appear many missing conditions, unlike the standard textbook questions where all the information are given precisely. Through demonstrating the processes involved in handling the above task, the processes involved in the modeling processes will be presented.

Step 1: Mathematisation

The modelling task is about making profits in a business transaction and appropriately fixing the selling price of a cake based on a given recipe.

In understanding profit and selling price (and cost price) of transaction mathematics, students will soon realize that the problem lies in the mathematical equation

Profit = Selling Price – Cost Price.

To fix the selling price, students need to recognize that they are required to (i) work out the cost price (based on the given information about the recipe) and (ii) decide on the profit for each cake.

Thus the first step of solving the above modelling task is to formulate the real-world problem into a mathematical one – mathematisation.

Step 2: Involving in Mathematical Computation

In working out the cost price, one needs to work out the cost of (i) cocoa powder; (ii) flour; (iii) butter; (iv) sugar and (v) eggs needed to bake one cake (Figure 1). In order to collect these data, students could be engaged to find out estimates of these prices from the grocery shops in their neighborhood or from the webpages. This process of data collection could turn out to be a meaningful activity for their subsequent learning of mathematics, as data collection and interpretation is a useful aspect of statistics education.

Alternatively, teachers could provide some form of scaffolding for students involving the cost of the ingredients of the cake (an example of a scaffold is shown in Figure 2) if they want to focus more on the calculation of the cost.

Ingredient	Cost	Compare	Required amount	Cost of ingredients
Cocoa powder	1 kg costs ___	1 bag = 15 cups		
Flour	1 kg costs ___	1 bag = 15 cups		
Butter	250g costs ___	250g = 2 cups		
Sugar	1 kg costs ___	1 bag = 15 cups		
	12 eggs cost __			

Figure 2. A sample of price list of the ingredients required for baking a cake.

Once the data is given, students would begin calculating the cost price of the cake. Here, students are expected to be able to perform mathematics computations involving fractions and proportion to obtain a plausible answer for the cost price.

Step 3: Looking Back

Students at this stage would need to advise Beng Seng on the profit he would want to set. The calculation of the selling price would then be obtained from adding the cost price to the profit. It would begin that the students would arbitrarily decide on the profit for the cake. Teachers could invite their students to review the answers they have obtained in the real world context.

For example, students could be engaged in reviewing the assumptions made:

- Is their selling price based on their presumed profit realistic? Would people want to patronize Beng Seng's confectionery with this price? Would the selling price enable Beng Seng to 'survive', based on a sound estimation?

- If the rental of the confectionery shop needs to be taken into consideration, how would this affect the selling price of the cake?
- How will wastage be factored into consideration in developing the price of the cake?

For modeling tasks, the solver needs to interpret the solution and reflect on the practicality or reality of the answer or solution obtained for the task.

Knowledge Required for the Task

Generally, modeling tasks require students not only be proficient in the related mathematical knowledge, but also *contextual knowledge*, which is the knowledge of the real world in which this problem is grounded. In the above task, the knowledge required consists of the following:

1. Profit = Selling Price – Cost Price;
2. Multiplication of fractions and whole numbers;
3. Proportional calculation;
4. In running a business, one would want to gain profit;
5. Profit should be reasonable to both the consumer and the seller; and
6. Other factors might affect the selling price of a product (e.g. rental and wastage).

Items 4, 5 and 6 are the c*ontextual knowledge* associated with the above task on Beng Seng's delight. This is the unique feature of modeling tasks which clearly shows the relevance of mathematics to the real world.

Teachers' Scaffolding

There are likely to be difficulties (or 'blockages') that students might encounter in modeling tasks (Galbraith & Stillman, 2006; Maaβ, 2006). Students might have difficulty understanding the problems, or they might be limited by their mathematical knowledge. More importantly, students could also have difficulties making assumptions and identifying the key variables in modelling tasks, or they might be deficient in the contextual knowledge related to the tasks.

Thus, teachers should provide the appropriate scaffolding to guide the students through the task. We shall illustrate with examples on the significant role that teachers can play with reference to the task on Beng Seng's Delight.

1. Include More Information

In the above modelling task on Beng Seng's Delight, it is likely that students without much exposure to solving real world problems would find too much missing information.

Without understanding the problem, it would be difficult to advise Beng Seng on the selling price of the cake. At the initial stage of engaging students in solving real world problems, teachers might want to include more information to make the task manageable:

> Beng Seng opens a bakery shop in Orchard Road. He has a secret recipe shown above for his famous chocolate cake. Advise him how much he should fix the selling price of a Beng Seng's Delightful Choco Cake <u>if he wants to have a profit of $20 per cake.</u>

In the original task, although it would be desirable to leave to the students to collect data on the cost of ingredients (for their exposure to data collection), a table consisting of the cost of the various ingredients as in Figure 2 would be useful for beginning students. This could help the students to focus on the *mathematization* part of the task to be completed.

2. Help Students with the Mathematical Knowledge for the Task

Tasks involving real world applications usually consist of mathematical knowledge from more than one topic as shown in the above example. It is thus important that teachers provide the students with appropriate scaffolding when their students are "stuck" with a particular concept.

It is strongly encouraged that teachers do not immediately respond to students by providing them with the answers or the direct mathematical steps leading to the required answers of the task; rather, the teachers' role in scaffolding should be likened to a laboratory technician's role in the science laboratory in helping to fix the apparatus for the scientist to work with the experiment rather than taking over the role of the scientist to carry out the experiment! Teachers must be clear that any help provided for the students should be to facilitate them solve the mathematical problems.

When students encounter difficulty with the problem, modifying the ideas of scaffolding from the Mathematics Problem Solving (Toh, Quek, Leong, Dindyal & Tay, 2011), different levels of help can be provided:

Level 1: generic help;
Level 2: specific help; and
Level 3: answering specific part of the problem.

At Level 1 help, students are guided to relate to the mathematics behind the calculation themselves. Some examples of level 1 help are shown below.

- *How do you calculate the profit of the cake?*
- *How is the profit related to the selling price?*
- *With all these ingredients (for the cake), how do you calculate the cost price of a cake?*
- *What information do you need to calculate the cost price of the cake?*

Level 2 help reaches out to students who are stuck with particular mathematical sections that the teachers have identified. Examples of level 2 help are shown below.

- *How do you multiply mixed numbers with whole numbers?*
- *If 12 eggs cost $2, how much will 4 eggs cost?*

Teachers are reminded that it is advisable to begin with Level 1 help whenever students are stuck. This will help them with understanding and identifying the real world constraints related to the task.

3. Facilitate the Students to Reflect on Their Answers

Looking back on one's solution is an important stage of mathematical problem solving. This is especially important for modeling, in which a key component is to connect the mathematical solution back to the real world. Thus, students must make sense of the solution in the real-world context.

Teachers could invite the students to think over their answer, as in:

- *Now that you have fixed your selling price to be XX, do you think it is reasonable based on the profit that you advise Beng Seng?*

Teachers could even challenge their students to change their assumption for the task:

- *Now, what if you take the rent of the shop into consideration?*
- *How would you factor into consideration the selling price if everyday there is 10% of the cakes which could not be sold?*

Conclusion

We have discussed how application of mathematics in the real world could begin with using context to a modelling task. Teachers are reminded that the whole idea as discussed above is to allow students to connect classroom mathematics to the real world in order to show the applicability of mathematics ideas (Zbiek & Conner, 2006; Stillman, 2010).

To begin with, teachers could begin with using relevant examples of real-world problems from textbooks, and emphasize that in many of the real-world problems, the numbers and answers are not 'neat', nor are the answers unique. Furthermore, lead students to be aware that in many real-world problems, the correctness of the answers is dependent on the assumptions made.

Modelling tasks involve both mathematical content knowledge and contextual knowledge. Thus, it is also important that teachers are proficient with both content and contextual knowledge required for carrying out such tasks. In addition, the pedagogy and scaffolding are important aspects for teachers conducting such activities in real-world problems.

References

Ang, K. C. (2009). *Mathematical modeling and real life problem solving*. In B. Kaur, B.H. Yeap & M. Kapur (Eds.), Mathematical problem solving: AME Yearbook 2009 (pp. 159 – 182). Singapore: World Scientific.

Galbraith, P., & Stillman, G. (2006). *A framework for identifying student blockages during transitions in the modelling process*. ZDM, 38(2), 143–162.

Maaβ, K. (2006). *What are modelling competencies?* ZDM, 38(2), 113–142.

Ministry of Education. (2006a). *Mathematics syllabus: Primary. Singapore*: Author.

Ministry of Education. (2006b). *Mathematics syllabus: Secondary. Singapore*: Author.

Stillman, G. (2010). *Implementing applications and modeling in secondary school: issues for teaching and learning.* In B. Kaur & J. Dindyal (Eds.), Mathematical applications and modeling: AME Yearbook 2010 (pp. 300 – 322). Singapore: World Scientific.

Toh, T. L., Quek, K. S., Leong, Y. H., Dindyal, J., & Tay, E. G. (2011). *Making mathematics practical: An approach to problem solving.* Singapore: World Scientific.

Toh T. L., Lim Y. C., Chua J., & Heng J. (2006). *Mathematics matters for normal* (Technical) Sec 1. Singapore: EPB.

Williams, D. L. (2007). *The what, why, and how of contextual teaching in a mathematics classroom.* Mathematics Teacher, 100(8), 572–575.

Zbiek, R. M., & Conner, A. (2006). *Beyond motivation: Exploring mathematical modeling as a context for deepening students' understanding of curricular mathematics.* Educational Studies in Mathematics, 63, 89–112.

SELECTION OF PROBLEMS FOR A PROBLEM SOLVING MODULE: FROM A SPECIALISED SCHOOL TO THE MAINSTREAM SCHOOLS

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We have presented the conceptualization of a mathematics problem solving module (Toh, Quek & Tay, 2008). In designing the module, mathematically rich problems were selected to enable students to acquire the various aspects of problem solving processes based on Polya's stages. At the initial stage when the module was taught in a Singapore secondary school which specializes in mathematical sciences, effort was taken to ensure that the problems were sufficiently challenging to engage the students to explore the problem solving processes. As the module was subsequently introduced to other mainstream schools, the choice of problems was re-considered. This paper discusses how the choice of the problems changed from the initial phase of the research project to the stage of its diffusion to the mainstream schools.

Introduction

We have described the conceptualization of a problem solving module in a Singapore school (Toh, Quek & Tay, 2008). The problem solving module was developed as part of a research project carried out by the authors (hereafter "researchers"). The choice of problems is a critical part in enacting a problem solving curriculum (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). The problems selected must be mathematically rich which could allow students to apply the processes of mathematical problem solving.

In this paper, we describe how we select the problems for the problem solving curriculum in the first Singapore research school, a school specializing in mathematics and science, and how, in the context of a design experiment (Middleton, Gorard, Taylor & Bannan-Ritland, 2006), some of the problems were retained while others were replaced when this module was subsequently diffused to other mainstream schools in Singapore.

Mathematical Problem Solving for Everyone

The problem solving module was developed as part of the research project Mathematical Problem Solving for Everyone (MProSE). At its initial phase, the curriculum was conceptualized and implemented in the first research school (hereafter we call it MProSE research school). This school which specializes in mathematical sciences was used to pilot MProSE in an attempt to use the "best-case scenario" method to start our investigation with high-ability mathematics students in the school. The researchers believed that the test bed for the initialization phase of an innovation should be at the school that is most conducive for success.

MProSE uses the design experiment as its methodology. This methodology appealed to the researchers in that it allows for the unique demands and constraints of the schools to be met, and, at the same time, the research imposes rigour on the design. The methodology's advocacy of an implement-research-refine iterative approach to educational design holds potential in dealing with the complexity of school-based innovations.

Selection of Problems for the Curriculum

The selection of problems for this module formed an important process of designing this module as each of the problems selected for this module is used to illustrate the various aspects of problem solving based on Polya's model at various juncture of the module. At the initial stage of selecting the problems, the researchers recognized that these problems must, in addition be mathematically "rich", elicit the problem solving processes that are essential in handling non-routine problems. In addition, the following four points of reference were also taken into consideration:

1. the problems were interesting enough for most if not all of the students to attempt the problems;
2. the students had enough "resources" to solve the problem;
3. the content domain was important but subordinate to processes involved in solving it; and
4. the problems were extensible and generalizable.

There are bound to be mismatches between what the assigner wishes to achieve and what actually is achieved during the solving process. Although the researchers were careful to ascertain if the problem satisfied the above four points of reference, it was recognized that these problems had not been tried out by the students yet. Feedback was obtained from both students and teachers of the MProSE research school after the first implementation in the MProSE research school. Some problems originally designed by the researchers were subsequently replaced (Quek, Toh, Dindyal, Leong, Tay & Lou, 2010). Eventually, an entire set of 17 problems was developed for the entire module consisting of ten lessons of problem solving in the MProSE research school (Toh, Quek, Leong, Dindyal & Tay, 2011a).

MProSE: Problems for the Mainstream Schools

It was found that the students from MProSE research school were generally able to demonstrate the use of Polya's model in solving mathematics problem (Toh, Quek, Leong, Dindyal & Tay, 2011b). This problem solving module was ultimately adopted as a compulsory component of the school's mathematics curriculum for all Year 8 (age 13 to 14) students of the school.

The usual mathematics curriculum in the MProSE research school stretches the students beyond the national mathematics curriculum as the school was not bounded by the Singapore national examinations. The students learn more sophisticated mathematical content knowledge even at the lower secondary level and are exposed to competition type mathematics questions. Furthermore, the students are generally highly motivated in mathematics and sciences. These students generally have more mathematical "resources" compared with their counterparts from the mainstream schools. Thus, the problem solving module which has worked in that school might not warrant its feasibility in the mainstream schools.

A problem solving seminar was organised with a view of inviting teachers from the Singapore mainstream schools to participate in the diffusion stage of the MProSE design experiment. The seminar disseminated the findings and shared lessons learnt from the MProSE research school. Generally, the feedback from the participating teachers from the mainstream schools was positive; the teachers from thirteen schools expressed their initial interest to participate in the MProSE project. Subsequently, the MProSE team obtained written forms of commitment of participation from four schools, roughly spanning across the whole spectrum of performance band, to implement the MProSE problem solving module at the lower secondary levels (age 13 to 14).

Problems Replaced for the Mainstream Schools

Prior to the commencement of the project, the researchers and the teachers from the participating schools met to discuss the details of MProSE, including the choice of problems for the schools. It was agreed that the design of MProSE was relevant to the mainstream schools, but some of the original MProSE mathematics problems (Toh et al, 2011a) must be replaced to meet the needs of these schools while other problems could be retained. In the following subsection, we discuss how the set of MProSE problems evolved as MProSE was diffused to three mainstream schools.

Content Not Emphasized in the School Curriculum

The following MProSE problems that were developed in the initial phase for the MProSE research school, with the heuristics that each problem is supposed to serve listed below the problem statement, were replaced.

Problem 2:	Find the last digit of 13^{77}. *Heuristics:* Consider a simpler problem; Make a systematic list; Look for patterns.
Problem 3:	Find the last digit of $1962^{2009} + 2009^{1962}$. *Heuristics:* Think of a related problem; Aim for subgoal.
Problem 6:	Show that the integer n always has the same last digit as its fifth power. *Heuristics:* Use suitable numbers; Think of a related problem; Divide into cases.

These "last digit" questions were manageable by practically all students from the MProSE research schools. Through solving these questions, the elements of Polya's problem solving stages could be brought across to students reasonably clearly. These problems were typical competition type questions; the content knowledge underlying these two problems is not emphasized in the mainstream Singapore secondary mathematics curriculum. The teachers from the mainstream schools felt that these problems were generally not suitable for their students as they generally have not been exposed to such questions on number theory at the lower secondary school level. In addition, both the teachers and the MProSE researchers agreed that the structure of the problems for the problem solving module should not be radically different from the questions from the usual school curriculum. Consequently, the researchers replaced the three problems with the following, with each problem emphasizing the same heuristics as the original corresponding problems.

Problem 2 (New):	Simplify $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{100\times 101}$.
Problem 3 (New):	Find the sum of the positive odd numbers 1 + 3 + 5 + 7 + …….. + 2011. Justify your answer.
Problem 6 (New):	On a Tuesday, Alice, Bernice and Carol and Dory, met for a movie. After the movie, they made plans for the next gathering. Alice, Bernice and Carol said that they could only go to the movies every 6, 3 and 4 days respectively, starting from that Tuesday. Dory said that she could go to the movies every day except on Sundays. After how many days would the four friends be able to meet again for a movie?

Compared with the original Problem 2, the new Problem 2 is not totally unseen by the mainstream school students as this question (or a simplified version of this problem with 10 terms instead of 100 terms) has likely been used in the primary school as an enrichment activity on the use of calculators.

Similarly, the new Problem 3 was adapted from a mainstream school examination for lower secondary school. Instead of providing the scaffolding to lead students to observe the patterns as in the question from the examination, the solver is expected to work out the heuristics themselves.

The new Problem 6 was adapted from a practice question from a Singapore mathematics textbook, with the exception of including an additional condition (that Dory could not go for movies on Sundays) which requires students to check the given condition.

Content Too Advanced for General Student Population

In adapting problems to the mainstream schools, the problems for which the content was generally beyond the grasp of the general student populations were replaced by questions from which the content is within the secondary school curriculum.

Problem 11:	The base 2 representation of a positive integer n is the sequence $a_k a_{k-1} a_{k-2} \ldots . a_0$, where $n = a_k 2^k + a_{k-1} 2^{k-1} + \ldots . + a_1 2^1 + a_0$, and $a_k = 1$, and $a_i = 0$ or 1 for $i = 0, 1, \ldots ., k - 1$. Write down the binary representation of the day today and of the day of the month in which you were born. *Heuristics:* Use numbers; Consider a simpler problem; Look for patterns.
Problem 12:	Weights can be placed on the left pan of a standard two-pan balance to weigh gold which is placed on the right pan. Suppose we want to be able to weigh gold in any positive integer up to 100 grams. Show that having 7 weights will be enough. *Heuristics:* Act it out; Use numbers; Consider a simpler problem; Look for patterns; Think of a related problem

Problems 11 and 12 were considered to be too difficult for mainstream schools, as their students do not encounter binary representation of numbers. These two questions were replaced with the following problems, emphasizing the same heuristics as the corresponding original problems.

New Problem 11:	Is it true that the sum of two rational numbers is always rational? Justify your answer.
New Problem 12:	Can you construct a circle where both the circumference and the diameter are integer lengths? Justify your answer.

These two new problems focus on integers and rational numbers, and also serve to enhance students' understanding of school mathematics in addition to problem solving.

Context-free Problem

Consider the following MProSE problem on curve sketching.

Problem 10:	It is given that $\lvert x \rvert = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.
	Sketch the graph of y = 3|x| - 4.

The concept on absolute value function is not emphasized in the lower secondary mathematics curriculum. The teachers felt that the content would be beyond the grasp of their lower secondary students. From the researchers' perspective, absolute value function was only used as a context to illuminate the heuristics of using suitable numbers (an important heuristic) in plotting a graph. This would provide an alternative way to learn curve sketching to memorizing the shapes of "standard graphs".

On the other hand, this question might be "meaningless" to students, as they do not learn much curve sketching at the lower secondary level. More importantly, this problem is context-free; its esoteric nature might not bring across the relevance of mathematics to students. Mathematics teachers have the common experience that their students want to know the relevance of classroom mathematics to their lives. Engaging students in solving real-world mathematics problems can thus help to give students the meaning to the mathematical content knowledge that they learn (Williams, 2007).

To compromise between the teachers' concern and the researchers' emphasis on the important problem solving skills for sketching graphs, a contextualized question involving curve sketching was introduced for Problem 10.

Problem 10 (New):	In a carpark, it was advertised as such: *First hour parking: Free* *Subsequent parking: $1 per hour or part thereof* Sketch a graph for parking fee versus the number of hours of parking. (Adapted from Toh, 2010).

It is clear that sketching the graph in the new Problem 10 requires the same heuristics of using suitable numbers and plotting some points as the original Problem 10.

Problems that were Retained

In the process of adapting the problems for the mainstream schools, many of the remaining 17 problems were retained. We shall discuss the attributes of three of these problems.

Problem 1:	You are given two jugs, one holds 5 litres of water when full and the other holds 3 litres of water when full. There are no markings on either jug and the cross-section of each jug is not uniform. Show how to measure out exactly 4 litres of water from a fountain. Show also the following: i. Get 2 litres from 3 litre and 7 litre jugs. ii. Get 6 litres from 12 litre and 16 litre jugs. iii. Get 12 litres from 18 litre and 24 litre jugs.

It was agreed that this problem is interesting to students as it vividly describes a task that students can associate with in real life. Further, the statement of the problem is not too difficult for students to understand.

This problem is illustrative in the application of the heuristics of *drawing a diagram, acting out, guess-and-check,* and even formulating the problem *using an algebraic equation.* Further, the statement of this problem shows the importance of *understanding the problem* (viz. the significance of the statement that "there are no markings on the jug and the cross-section is not uniform).

The feedback from the mainstream school teachers after initialization of problem solving module was that the students were generally excited over this problem. In fact the problem was taken from a scene in the movie "Die Hard with a Vengence" where the protagonist had to measure exactly 4 litres of water to be placed on a sensor to prevent a bomb from going off. The engagement level of students who were shown this scene were very high.

Problem 4:	Two bullets are placed in two consecutive chambers of a 6-chamber revolver. The cylinder is then spun. Two persons play a safe version of Russian Roulette. The first points the gun at his mobile phone and pulls the trigger. The shot is blank. Suppose you are the second person and it is now your turn to point the gun at your hand phone and pull the trigger. Should you pull the trigger or spin the cylinder another time before pulling the trigger?

The context of this problem excites the students to solve this problem. Students have already acquired intuitive idea of chance even in primary school mathematics curriculum, although they might not have learnt the formal definition of probability. This problem could be solved by the heuristics of *drawing diagrams* and *considering different cases.*

This problem managed to captivate the interest of students of both the MProSE research school and the mainstream schools alike in that it involves the students' making decision in an interesting context. However, there was a slight concern raised by the teachers that some students were not able to solve this due to their lack of the related *contextual knowledge* (that is, knowledge of how a revolver functions).

Problem 13:	a) A boy claims that when he left school X and joined school Y, he raised the average IQ of both schools. Explain if this is possible. b) A striker in football is rated according to the average number of goals he scores in a game. Wayne had a higher average than Carlos for games in the year 2007. He also had a higher average than Carlos for games in the year 2008. Can we say that Wayne must be a better striker than Carlos over the years 2007-2008?

This problem is exemplary in demonstrating the heuristics of substituting suitable numbers and working backwards. The problem forces the students to think of actual numerical examples to support their claims. Furthermore, both the content and the context of this problem are relevant to them. It further challenges the students' intuitive idea of statistical averages and encourages them to think more deeply about their knowledge of statistics.

Attributes of Problems Used in Mainstream School

From the researchers' work of adapting MProSE problem solving module to the mainstream schools, the problems that were used in the mainstream schools fall under one of the following categories:

- The problems are concrete and could engage students in some sort of hands-on activities (problems 1 and 4);
- The context of the problems involves students' interest or immediate surrounding (new problem 10, problem 13);

In addition, the mathematical content knowledge of the problems used in the MProSE programme is not too far-fetched from the national school mathematics curriculum.

Conclusion

In an effort to introduce the innovative MProSE problem solving from a specialized school to the mainstream schools, the researchers have retained or replaced the mathematical problems discussed above. Taking note that since the MProSE program highlights teaching *about* problem solving, the researchers were mindful that the selected problems were subordinate to the problem solving processes that are expected to be developed in the students. Thus, the problems could be replaced with those whose content, context and level of difficulty suitable for the students. As the MProSE research project evolves and diffuses to more mainstream schools, it is certain that the mathematical problems will be further modified or adapted to meet the needs of different audiences.

References

Middleton, J., Gorard, S., Taylor, C., & Bannan-Ritland, B. (2006). The 'compleat' design experiment: From soup to nuts. *Department of Educational Studies Research Paper 2006/05 University of York.*

Quek, K. S., Toh, T. L., Dindyal, J., Leong, Y. H., Tay, E. G., & Lou, S. T. (2010). Resources for teaching problem solving: A problem to discuss. In L. Sparrow, B. Kissane, & C. Hurst, (Eds.), *Shaping the future of mathematics education* (pp. 753-756). Fremantle, Australia: The Mathematics Education Research Group of Australasia.

Silver, E., Ghousseini, H., Gosen, D., Charalambous, C., & Strawhun, B. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal for Mathematical Behavior,* 24, 287–301.

Toh, T., Quek, K., & Tay, E. (2008). Mathematical problem solving - A new paradigm. In J. Vincent, R. Pierce, & J. Dowsey (Eds.), *ConnectED maths. Proceedings of the 45th Annual Conference of the Mathematical Association on Victoria.* (pp. 356 - 365). Melbourne: MAV.

Toh, T., Quek, K., Leong, Y., Dindyal, J., & Tay, E. (2011a). *Making mathematics practical: An approach to problem solving.* Singapore: World Scientific.

Toh, T., Quek, K., Leong, Y., Dindyal, J., & Tay, E. (2011b). Assessing problem solving in the mathematics curriculum: A new approach. In K. Wong, B. Kaur (Eds.), *AME Yearbook 2011: Assessment* (pp. 1-35). Singapore: World Scientific.

Williams, D. (2007). The what, why, and how of contextual teaching in a mathematics classroom. *Mathematics Teacher, 100*(8), 572–575.

MATHEMATICA™ NOTEBOOKS AS PEDAGOGICAL AND ASSESSMENT TOOL

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Introduction

The Trial

The VCAA is conducting a small-scale three year trial (2011 – 2013) for computer-based delivery and student response to the extended answer component of Mathematical Methods (CAS) Examination 2 (HREF1). The computer algebra system (CAS) *Mathematica* is being used to deliver this component of the examination, as a computational tool, and for collecting student responses to this assessment. The trial is being conducted in seven schools across government, catholic and independent sectors and aims to develop and support effective alignment between the use of technology in curriculum, pedagogy and assessment. *Mathematica* (HREF2) can be used to develop interactive mathematical documents, called notebooks that integrate text with 'live' mathematical computations (calculations, tables, graphs, diagrams, symbolic expressions).This paper provides illustrative examples and discusses some implementation challenges and ways of addressing them.

Familiarity and Comfort

The introduction of a computer-based software platform for CAS technology in the classroom presents a significant change from the use of a hand-held calculator, and poses various challenges as well as benefits for teachers and students alike. As with anything new or different there will naturally be apprehension and anxiety amongst students and teachers. To successfully manage such a transition this must be explicitly acknowledged and addressed. Empowering students with a basic understanding of *Mathematica* enables them to overcome their apprehension and anxiety and use it effectively for working mathematically in the classroom and in technology enabled assessment. Based on experience of two trial school teachers, we outline some approaches for facilitating the transition process by considering the issues of: managing perceptions; diagnosing common problems; creating and utilising assistance sheets; and the transition to student development of their own notebooks, for example, summary notes.

Mathematica is a powerful and flexible tool that is widely used in industry, academia, business and finance, and research as well as in education and there is a wide community of users and applications. However many students are unlikely to be familiar with such software and the style of interface.

A Positive Motivation

Where students are already familiar with handheld CAS devices, they will be wary of switching to computer based CAS software. As an educator it is important to put forward the benefits of such software both by discussion and demonstration. For computer based CAS software benefits include a large screen and good visual representation; easy keyboard and palette input; and powerful and quick processing/computational capability. The ability to cut and paste and readily access the usual desktop environment functionality is also convenient. Students will appreciate these but nonetheless still need to know how to readily apply commands such as **`Solve`** and **`Plot`**. If students have not previously used a CAS calculator, this process will be 'new' for either technology. If they have used a CAS calculator previously they will find aspects of the command interface similar.

As a simple example, students using a hand-held device would employ a combination of menu selection and keypad entry to do something like the computation shown in Figure 1.

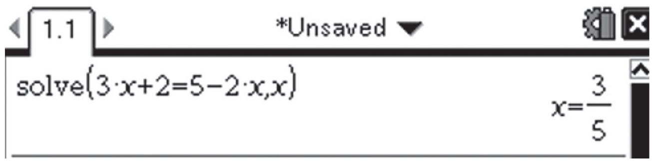


Figure 1. Sample hand-held computation

Using *Mathematica* the student could carry out the same computation by typing in the required expression directly and evaluating via the keyboard (Shift+Enter) or alternatively use a palette for entry and then evaluating as shown in Figure 2.

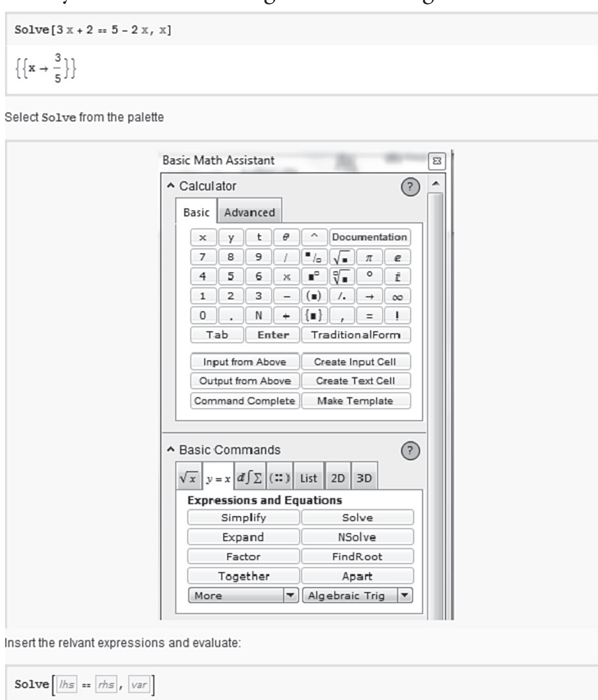


Figure 2. Sample Mathematica computation

The Quick and Easy Approach

An aspect that will attract interest (once students are familiar with it) is the easy approach to defining and working with a function. This can be done on handheld devices, but tracking and changing is convenient using the software. Once a function is defined it can be called by name and used in various ways as shown in Figure 3.

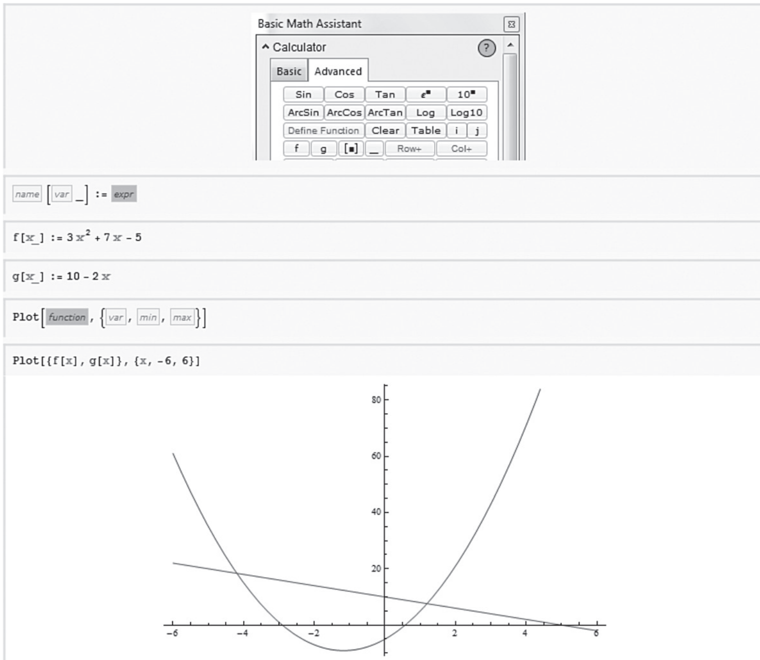


Figure 3. Defining two functions and drawing their graphs on the same set of axes

Related equations can be readily solved, as shown in Figure 4.

```
Solve[lhs == rhs, var]
```

```
Solve[f[x] == g[x], x]
```

$$\left\{\left\{x \to \frac{1}{6}\left(-7-\sqrt{253}\right)\right\},\ \left\{x \to \frac{1}{6}\left(-7+\sqrt{253}\right)\right\}\right\}$$

```
N[Solve[f[x] == g[x], x], 10]
```

$\{\{x \to -3.817662287\},\ \{x \to 1.484328953\}\}$

Figure 4. Solving for the x coordinates of the points of intersection of the graphs of f and g

Existing expressions can be readily copied, edited and re-evaluated, as shown in Figure 5, which is based on the last evaluation in Figure 3, editing the function and including an axes labelling option:

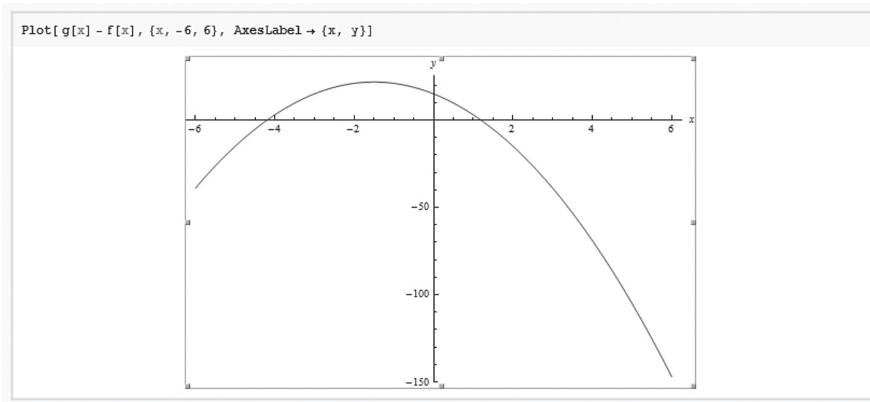


Figure 5. Plot of g(x) – f(x) evaluated after editing earlier computation

Manipulating a Situation

Mathematica is not just a powerful software version of a 'desktop calculator' although it can certainly be used in this way as shown in Figure 6.

```
N[e^20, 1000]

4.851651954097902779691068305415405586846389889448472543536108003159779961
  1427097401659798506527473494478337894389613263578237003165528236452639 1
  5628556525364008013517605841904211694138315421807381039587198978360763 2
  0595581385884205559608561666290292161301767529853538672630870988601710 85
  9299413082880839412345923599971661197400001781308413023075165235880979 6
  2258856354033937319471822537909832696775124258966471236150772660867099 6
  3980932916820481764485350032757096687202805087705277026052199157940102 3
  5397991822950931688767399180286336188612912955613171131406176075211472 1
  4251360376454606175236700346541442291547206283650835208059298365802457 5
  5622602227151222481334887821292624112223086348377249196676510834689333 4
  5119308311963893153811081385130903423017597168194310660403667300517049 3
  3356145701889061738002402568527920476033636709826830133275937548591727 5
  5095994290289610395770292639009843371726696821373003107326203011459793 0
  8694695778860181722726393624831241674430552210132142713351291495438146 8
  30234 × 10^8
```

Figure 6. Evaluation of e20 to 1000 significant figures.

A useful way to introduce students to a broader view of its use is to work with previously developed notebooks (in which computations can be readily edited and re-evaluated) to illustrate the behaviour of a family of graphs. As a simple example consider a family of graphs of the form $y = mx + c$. Using the **Manipulate** functionality one can readily illustrate the effect of varying m and c and compare it with the graph of $y = x$, as shown in Figure 7.

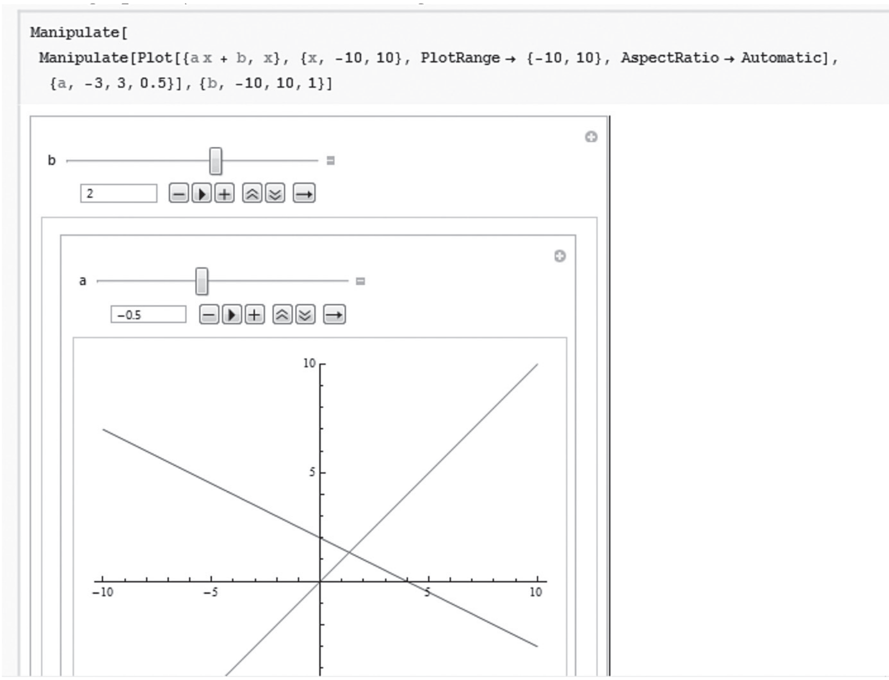


Figure 7. Dynamic exploration of linear graphs using **Manipulate**

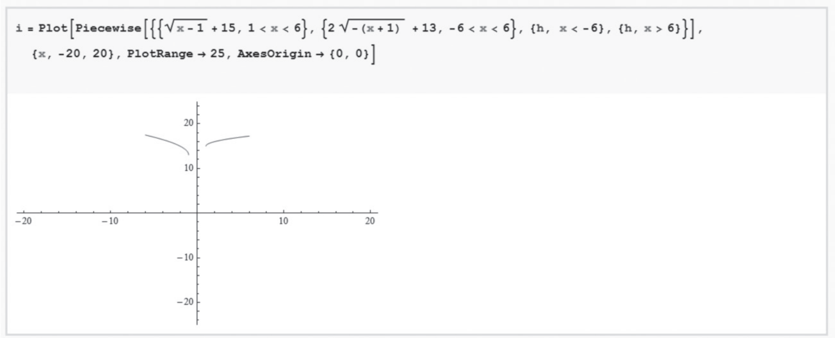
Diagnosing Common User Problems

The first half year of using *Mathematica* from scratch in a transition context in the classroom can be fairly challenging, especially when students are unfamiliar with the use of software as enabling technology, and have already developed some facility with calculators. Students feel hesitant and the teacher is likely to be in the process of developing their own confidence and familiarity with using the software. While a teacher can nicely demonstrate judiciously chosen examples during whole-class instruction, trouble shooting for students "why isn't mine working?" can initially be overwhelming. Even minor errors nonetheless affect or prevent a desired computation from taking place, and these are by their very nature not always easy to see and diagnose. After a while, the teacher starts to more consistently use the built in *Mathematica* colour key to troubleshoot and becomes better at resolving student problems on the spot. The second year of introducing students to *Mathematica* proceeded much more smoothly. Students were taught how to trouble-shoot their own problems from the start, resulting in student confidence and a general "hey *Mathematica* is not bad" ambience. The following are a few ideas on teaching students to trouble-shoot.

Use of Palettes

Mathematica palettes provide an easy way to use correct syntax. During the initial half year of introducing the software one teacher suggested students utilise *only* the palettes to input commands. While convenient for some, this resulted in less attention to learning required basic syntax, and hence a less robust and independent student trouble-shooting capability. With the second group of students the same teacher introduced *Mathematica* with a *combination* of palettes and direct keyboard input. The teacher would give examples on the board where the palette options were shown, but typed in the actual input for evaluation. This gave students the option of *choosing* which approach to use as applicable. Less confident students tended to use the palettes almost exclusively and found trouble-shooting more challenging, they were also less inclined to use *Mathematica* for computation and often used by hand computation instead. The majority of other students generally preferred to type input for evaluation directly in most cases, rather than rely on the palettes, and referred to the palettes mainly when they had forgotten the relevant syntax. These students quickly picked up on the specific uses of brackets, commas, operations and the like and could trouble-shoot with confidence.

Palettes can also be a vehicle for students to explore aspects of what *Mathematica* has to offer, for example, different options for `Plot`, such as including grid lines or graphing with thick dotted curves. One group even went as far as setting up gridlines so that each line was a different colour. Fun projects such as having students draw a colourful smiley face or sunset provides a useful context for them to investigate all the options within the palettes on their own, and enhances student understanding of the basic syntax structure. This utilises combinations of a variety of graphs of linear and non-linear relations, Figure 8 shows the 'eyebrow' component and final image for a 'smiley face':



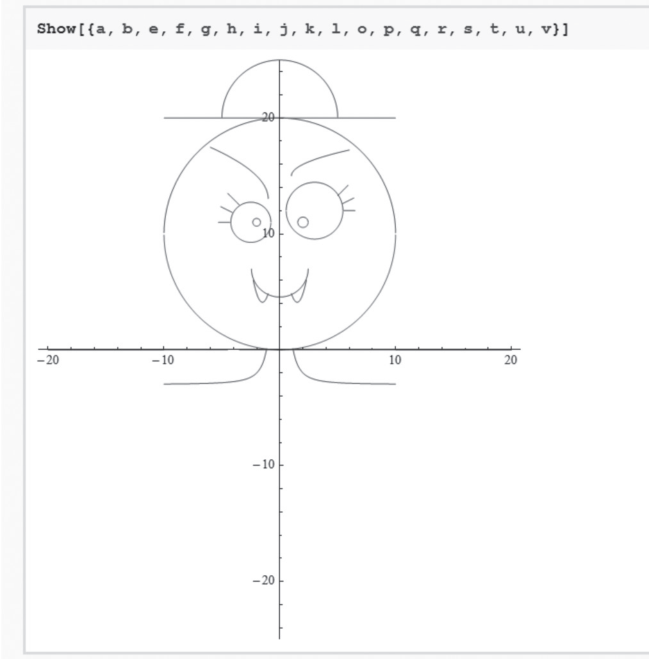


Figure 8. Graphs of 'eyebrows' as part of a 'smiley face'

Colour Diagnostics

While it is difficult to demonstrate colour changes within a black and white print environment, it is nonetheless helpful to note some of the key colour diagnostics available in *Mathematica*. To begin with, any predefined commands, such as **`Plot`** or **`Solve`**, will be written in black. One way to highlight this to students is to start by typing "**`Plo`**" in

Mathematica. This will come up in *blue* typeface. Once the final "`t`" is written, the blue `Plo` will turn into a black `Plot`. This is useful to point out to students when they are starting to assign variables. Here are two examples of what a student may type to solve $x + y = 2$ and $2x + 3y = 1$ simultaneously.

```
Solve[{x + y == 2, 2 x + 3 y == 1}, {x, y}]
{{x → 5, y → -3}}

Solve[{x + y == 2, 2 x + 3 y == 1}, {x, y}]
Solve::ivar : 3 is not a valid variable. ≫
Solve[{3 + x == 2, 9 + 2 x == 1}, {x, 3}]
```

The inputs from both examples are identical, and yet the solutions are different. What one doesn't notice as a beginning *Mathematica* teacher is that the student has, several lines previously, and hence no longer visible on their screen, defined y as 3. Thus, for the second computation, *Mathematica* is trying to solve a set of equations for which one of the variables is a constant. The error could have been avoided had the student noticed while typing in the equations, y did not appear the unassigned blue colour, but in black. What makes these issues so difficult is that once the user has specified y to be a variable, it will turn from blue/black to green. Thus when the entire `Solve` command is completed, both Solves really are identical, even down to the colours.

Taking note of whether a letter changes from *black/blue* to *green* when specified as a variable can help avoid errors such as

```
Solve[2 a == 4, x]
{}
```

The variable in the equation to solve is `a`, yet *Mathematica* has been told to solve for the letter ***x*** and we are given the empty set as the solution. In colour the letter ***a*** appears in *blue* while the letter ***x*** appears in *green*. Students should be taught from the start when using commands such as `Solve` and `Plot` requiring variable specification that variables need to be *green*.

This can help with the common problem when students begin to use CAS of forgetting to include spaces or symbols between two letters to represent implied multiply. Below are two attempts at trying to define the quadratic **ax²**. In the first attempt, a space is included between the ***a*** and ***x***. The ***a*** is coloured *blue* and the x is coloured the same *green* as the ***x*** within `f[x_]`. In the second attempt, there is no space between the ***a*** and the ***x***. Both the ***a*** and ***x*** remain *blue* while the ***x*** within `f[x_]` is *green*.

$$f[x\_] := a\ x^2$$

$$f[x\_] := ax^2$$

There is a range of useful diagnostic tools, however they do require some attention to be properly utilised. It is helpful to become familiar with using the built in **Documentation Center** which enables one to make a query and have a reference provided from a digital book, along with example evaluations which can be copied directly into an existing notebook and used, as shown in Figure 9:

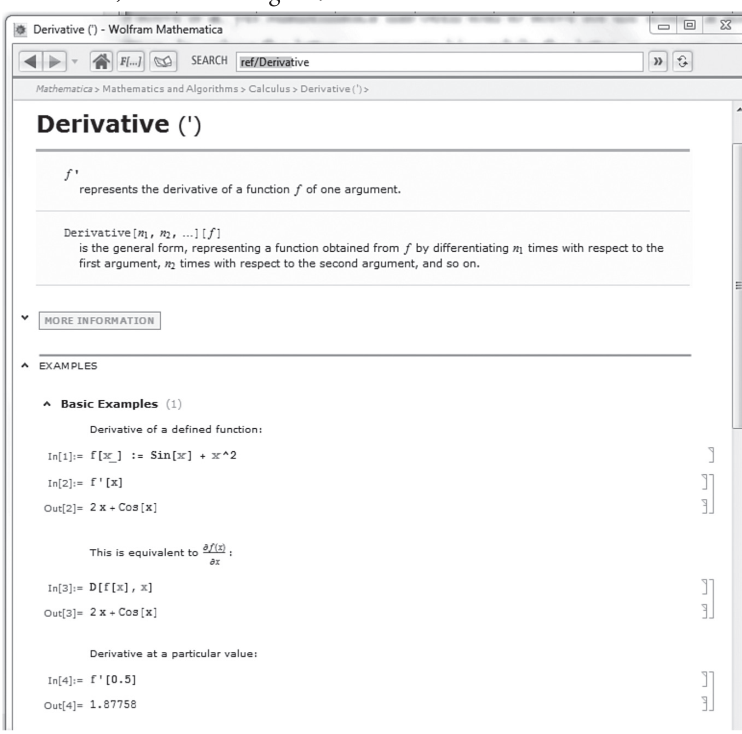


Figure 9. A Documentation Center query

Use of *Mathematica* in the Junior Year Levels

As more and more students have access to their own laptops in the classroom, *Mathematica* can offer a smooth introduction to the world of CAS. In the first instance, students should start with a previously developed worksheet (as a notebook file) where only one or two new commands directly related to the mathematics being studied are incorporated and taught within the worksheet itself. In general, they should not simply be given a blank *Mathematica* screen and asked to use it as a computational tool. For example, a group of Year 7's completed a fractions task using *Mathematica* where students were required to learn that *Shift* + *Enter* would calculate an input and *Ctrl* + *ForwardSlash* would generate the fraction template–. The instructions for these commands were included within the task itself so that the learning of the technology was not separate and disconnected from investigating fractions. While the task was generally well received by staff, they did take some convincing to use it in the first place. Weaker students had difficulty reconciling the fact that *Mathematica* could do the computation for you and did not understand that the task was *not* asking them how to get the answer, but rather required them to *understand* what the answer meant.

Investigative tasks can easily be created due to the flexibility *Mathematica* gives you when writing your own programs. As part of a task on ratios, Year 8 students were asked to distribute water between four tanks according to certain ratios. An interactive notebook, as shown in Figure 10, was created in a *Mathematica* notebook enabling students to distribute the water so that they could electronically check the validity of their problem solving strategies. While designing these types of activities is the next step beyond basic familiarity, it does highlight the potential and freedom of what the software offers.

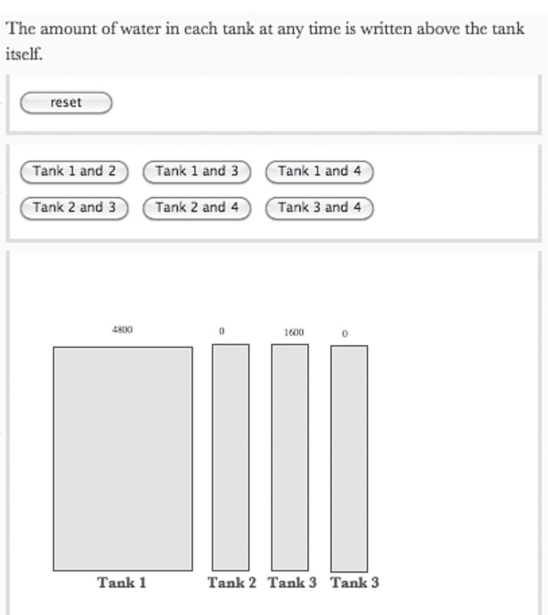


Figure 10. Teacher constructed dynamic activity in a notebook

Finding an Effective Balance

As *Mathematica* has many commands teaching only a limited set of relevant commands as required for the purpose at hand makes good sense (users can also generate their own palettes). A strategic approach when introducing commands is also important – *too few* and the students are limited in what they can do, *too many* and then they can become overwhelmed. There is no 'right' balance for all situations however what is effective in a given context will depend on the group, their level of competence, and the confidence and proficiency of the teacher.

Creating and Using Assistance Sheets (Student Notebooks)

Initially students may not see the value in creating their own assistance sheets, so teachers will need to provide examples and encourage students to progressively develop their own portfolio of *Mathematica* notebooks. On completing their first assessment task, or a practice task, then having an assistance sheet demonstrated by the teacher the attitude shift is noticeable. For technology enabled assessment students are allowed to freely use assistance sheets. The ability to readily develop a library of self-created commands, hints, examples and the like is a major benefit of CAS software such as *Mathematica*. The ease of access to such materials can be high-lighted as something that will save time and assist them greatly both in the class and for assessments.

A fundamental aspect of Mathematical Methods (CAS) Unit 1 is to understand the properties of polynomial functions and their graphs. An important skill is to be able to fit a suitable function to a set of data points. Using their understanding of functions, simultaneous equations and some other general skills students should be able to determine an equation that fits the data. Initially this should be done from first principles, however once students have learnt the basic processes and skills, they can then proceed to use the built in *Mathematica* functionalities. One such functionality **`Fit`** finds a linear combination of specified basic functions. It is important that the technology is not just used to obtain a result, but to explore the relationship between data and a model. For example, as part of an analysis task students might explore different modelling functions for a base-jumping context, as shown in Figure 11. The data points were supplied for students, with the image providing visual scaffolding for the problem.

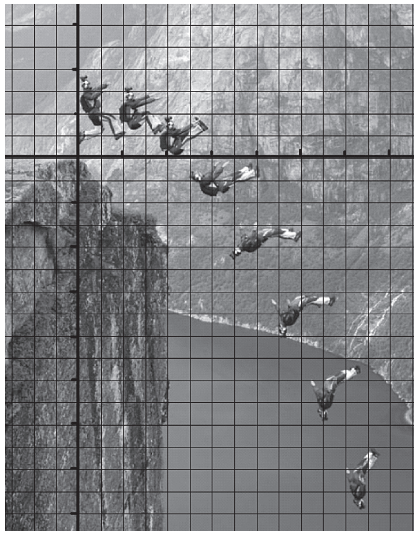


Figure 11. BASE – jumping

Once the data set has been defined it can be called within the **`Fit`** command and a function generated. A graph can be used to get a visual impression of goodness of fit, as shown in Figure 12.

```
basejumpdata = {{0.3, 1.7}, {1.1, 1.4}, {2, 0.7}, {2.6, -0.4}, {3.5, -2.1}, {4.6, -3.9}, {5.5, -5.7}}
{{0.3, 1.7}, {1.1, 1.4}, {2, 0.7}, {2.6, -0.4}, {3.5, -2.1}, {4.6, -3.9}, {5.5, -5.7}}
```

```
Fit[basejumpdata, {1, x, x^2}, x]
2.04033 - 0.49445 x - 0.170247 x^2
```

```
Show[ListPlot[{basejumpdata}], Plot[2.04032963750072` - 0.4944504535360087` x - 0.1702466795192717` x^2, {x, 0, 10}]]
```

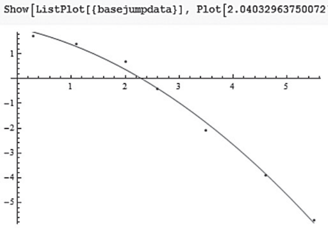


Figure 12. Graphical inspection of a model function

Analysis at Year 11 will generally be visual (or based on a simple numerical measure) and contrast this model with other simple possible function models such as a linear function or a cubic function. Summary statistics for goodness of fit can also be obtained using *Mathematica*.

References

Websites

HREF1: Victorian Curriculum and Assessment Authority. (2012). Conference Content retrieved from http://www.vcaa.vic.edu.au/Pages/vce/studies/mathematics/cas/castrial.aspx

HREF2: Wolfram Research. (2012). Conference Content retrieved from http://www.wolfram.com/

USING *MATHEMATICA*™ TO TACKLE MATHEMATICAL METHODS (CAS) EXAMINATION 2 MULTIPLE CHOICE QUESTIONS

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Brian Hodgson

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Introduction

Mathematica (HREF1) is computer algebra system (CAS) software that can be used as enabling technology to tackle various Mathematical Methods (CAS) Examination 2 multiple choice questions. A *Mathematica* notebook is an interactive digital document (file) that incorporates text, graphics and numerical, graphical and symbolic computations. They provide an excellent medium for an analysis summary of multiple choice questions by topic and type, across several years of past examination papers (2006 – 2011). This can be used to develop solutions by various methods and comment on aspects of the mathematics involved. Summary information about percentage of correct responses can also be included, along with hyperlinks to the VCAA website to access past papers and corresponding examination assessment reports (HREF2). These notebooks would be a valuable resource for student review and practice of past examination questions, and similar notebooks could also be developed for item response analysis tasks as part coursework assessment for Unit 4. *Mathematica* notebooks can be accessed by students even if they don't have access to *Mathematica* by means of the free CDF Reader from Wolfram Research (HREF3). Alternatively they can be converted to PDF format but do not have interactive text or computational capability in this format. Teachers and students could use these as a starting point to develop similar resources as e-documents for other CAS.

Structure

Four summary notebooks have been developed covering:

- functions, graphs and algebra;
- differential calculus and applications;
- integral calculus and applications; and
- probability

All the Examination 2 multiple choice questions have been allocated to one of these based on their principal focus, and copied into the relevant summary notebook as a PDF snippet. Each has been divided into a couple of key topics, and the type of question and year of paper identified. All past Examination 2 multiple choice questions have in this way been mapped into one of the four summary notebooks. This mapping is an activity that it would be useful for students to undertake themselves *before* being presented with a summary notebook or similar where this has already been done.

In this way they will likely strengthen their ability to carry out similar identification of questions during the reading time of the examination they actually sit. In the summary notebook, computations that support one or more solutions approaches and related commentary are shown, as applicable. The percentage of students selecting the correct response is also included, along with a summary of the relevant *Mathematica* functionality. Figure 1 shows the introductory section of the calculus anti-differentiation and applications summary notebook:

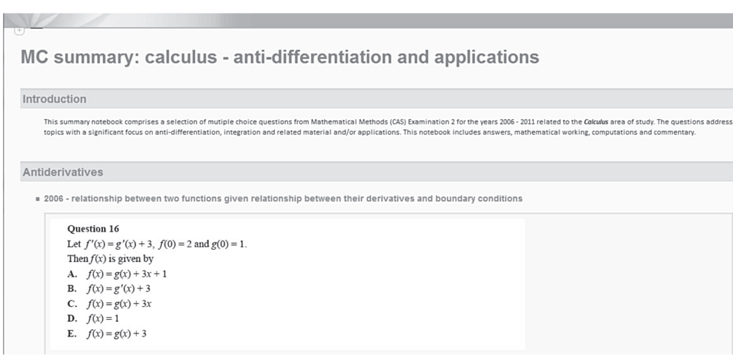


Figure 1. Introductory section of a summary notebook

Some Examples

Functions, Graphs and Algebra

The following discussion covers several examples from across the four summary notebooks, and indicates where the use of technology may be helpful. Consider Question 18 from the 2012 Mathematical Methods (CAS) Examination2, shown in Figure 2, for which 52% of students correctly answered (D):

▪ 2011 - interval over which a family of cubic functions has a unique zero

Question 18

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A. $-7 < w < 25$

B. $w \leq -7$

C. $w \geq 25$

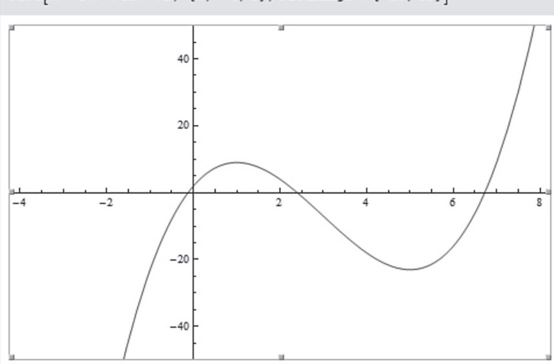
D. $w < -7$ or $w > 25$

E. $w > 1$

Figure 2. 2011 Examination 2 – multiple-choice Question 18

This question can be done without the use of technology – the graph of the corresponding function must have two turning points, so D is the only alternative with the correct form. However an empirical version of the same sort of thinking can be used, plotting graphs for judicious choices of w, for example $w = 2$, three horizontal axis intercepts eliminates A and E, while $w = 10$, one horizontal axis intercept and $w = 30$, also one horizontal axis intercept, eliminate C and B respectively, as shown in Figure 3:

```
Plot[x^3 - 9 x^2 + 15 x + 2, {x, -4, 8}, PlotRange → {-50, 50}]
```



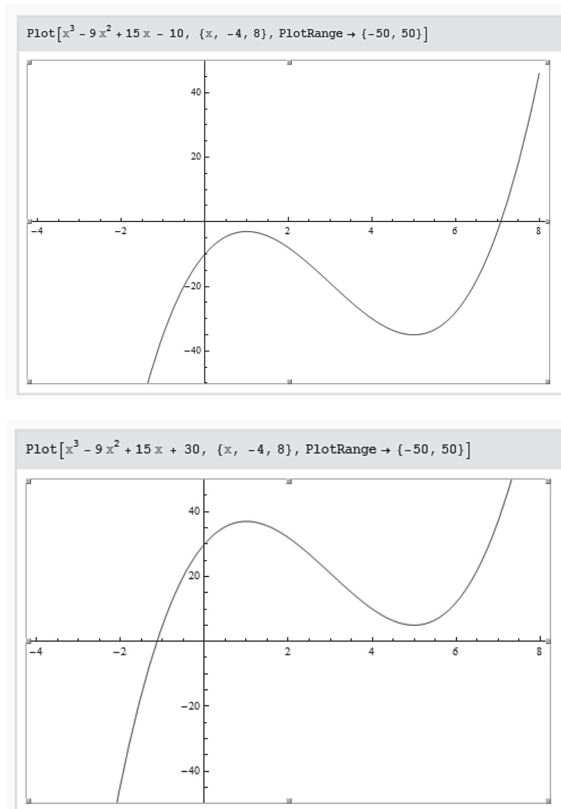


Figure 3. Graphs for $w = 2$, $w = -10$ and $w = 3$

An interesting variation on this analysis is to use the dynamic capability of the CAS and 'see' where the function has or doesn't have the required behaviour as w is systematically varied through integer values from -10 to 27, with the case for $w = 7$ shown in Figure 4.

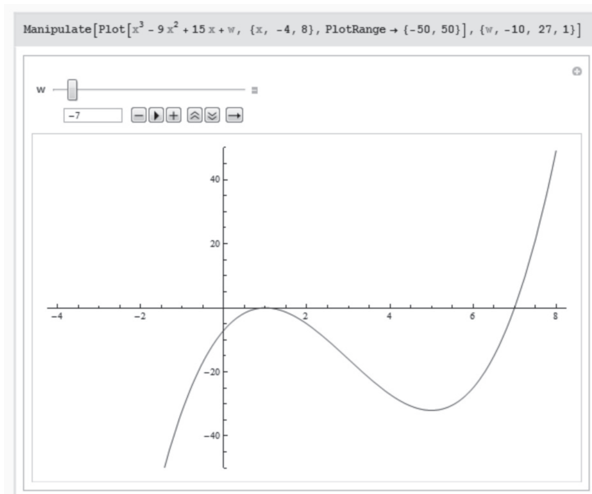


Figure 4. Mathematica dynamic manipulation

While not required to answer the question, the interval endpoints of -7 and 25 can be explicitly determined, as shown in Figure 5:

```
f[x_] := x^3 - 9 x^2 + 15 x + w

Solve[f'[x] == 0, x]
{{x → 1}, {x → 5}}

f[1]
7 + w

f[5]
-25 + w
```

Figure 5. Identifying endpoint values for w

Since **`f[1] = 7 + w < 0`** for one solution, **`w < -7`**. Similarly, since **`f[5] = -25 + w > 0`** for one solution, **`w > 25`**. The summary notebook also indicates relevant *Mathematica* functionality from built in palettes as shown in Figure 6.

▫ *Mathematica* functionality used for this question

```
Plot[function, {var, min, max}, PlotRange → {y min, y max}]

Manipulate[expr, control]

Solve[lhs == rhs, var]
```

Figure 6. Relevant Mathematica functionality from palettes

Commentary can also be included, as shown in Figure 7.

The answer can be deduced from the form of the question and the required form of the solution - **no computations are required.**

The graph of the function must have two turning points, otherwise the solution would be *R* (since the graph of a cubic must cut the horizontal axis at least once). Thus there will be only one solution when *w* is sufficiently large and positive (hence the local minimum is above the horizontal axis) *or w* is sufficiently large and negative (hence the local maximum is below the horizontal axis). D. is the only alternative of this form. Equality corresponds to two intercepts, for only one the interval must be open.

D (52%)

Figure 7. Commentary for Question 18

Probability

Consider Question 12 from the 2010 Mathematical Methods (CAS) Examination 2, shown in Figure 8, which 44% of students answered correctly (B):

▪ **2010 - discrete binomial cumulative less than**

Question 12

A soccer player is practising her goal kicking. She has a probability of $\frac{3}{5}$ of scoring a goal with each attempt. She has 15 attempts.

The probability that the number of goals she scores is less than 7 is closest to

A. 0.0612

B. 0.0950

C. 0.1181

D. 0.2131

E. 0.7869

B (44%)

Figure 8. 2010 Examination 2 – multiple-choice Question 12

This problem is modelled by a random variable X ~ Bi (15, 0.6) and the probability can be computed in several ways, from first principles, using the probability density function, and using the cumulative distribution function as shown in Figure 9.

```
Binomial[15, 0] 0.6^0 0.4^15 + Binomial[15, 1] 0.6^1 0.4^14 + Binomial[15, 2] 0.6^2 0.4^13 + Binomial[15, 3] 0.6^3 0.4^12 +
 Binomial[15, 4] 0.6^4 0.4^11 + Binomial[15, 5] 0.6^5 0.4^10 + Binomial[15, 6] 0.6^6 0.4^9

0.0950474
```

$$\sum_{n=0}^{6} \texttt{PDF[BinomialDistribution[15, 0.6], n]}$$

```
0.0950474
```

```
CDF[BinomialDistribution[15, 0.6], 6]

0.0950474
```

Figure 9. Three ways of computing the probability

Students were much more successful (71%) with a similar 'fair coin' question where $X \sim \mathrm{Bi}\,(12, 0.5)$ and $\Pr(X \leq 4)$ was required, possibly because the first principles computation was simpler for students who used this approach, the complexity of the other two methods doesn't really change.

Calculus

Related-rates are an area where students seem to be only moderately successful, for example, consider Question 10 from the 2006 Mathematical Methods (CAS) Examination 2, shown in Figure 10, which 48% of students answered correctly (A):

Rates of change and related rates

- 2006 - related rate (chain rule application)

Question 10

The radius of a sphere is increasing at a rate of 3 cm/min.

When the radius is 6 cm, the rate of increase, in cm^3/min, of the volume of the sphere is

A. 432π

B. 48π

C. 144π

D. 108π

E. 16π

Figure 10. 2006 Examination 2 – multiple-choice Question 10

While this is straightforward to compute as $V'(r) \times r'(t)$, as shown in Figure 11, some students may not have recognised that the second rate was a given constant rate, or possibly not have remembered the volume of a sphere formula correctly, or did not look it up from the formula sheet.

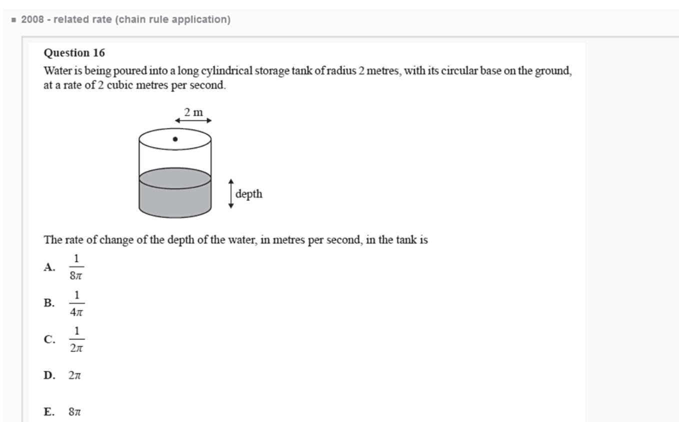
```
V[r_] := 4/3 π r^3

V'[6] × 3

432 π
```

Figure 11. Computing $V'(r) \times r'(t)$

Students did slightly better on Question 16 from the 2008 Mathematical Methods (CAS) Examination 2, shown in Figure 12, which 52% of students answered correctly (C):


2008 - related rate (chain rule application)

Question 16

Water is being poured into a long cylindrical storage tank of radius 2 metres, with its circular base on the ground, at a rate of 2 cubic metres per second.

The rate of change of the depth of the water, in metres per second, in the tank is

A. $\frac{1}{8\pi}$

B. $\frac{1}{4\pi}$

C. $\frac{1}{2\pi}$

D. 2π

E. 8π

Figure 12. 2008 Examination 2 – multiple-choice Question 16

This involves a constant rate and a reciprocal rate, where $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{dV/dh} \times \frac{dV}{dt}$ as shown in Figure 13.

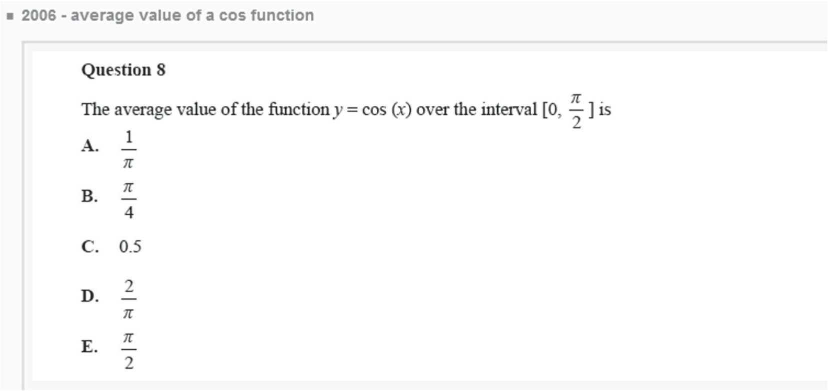
```
1/D[4 π h, h] × 2

1/(2 π)
```

Figure 13. Computing $\frac{dh}{dV} \times \frac{dV}{dt}$

For this question the answer can essentially be written down from mental computation.

The average value of a function over an interval is a simple application of the definite integral. Consider Question 8 from the 2006 Mathematical Methods (CAS) Examination 2, shown in Figure 14, which 59 % of students correctly answered (D). If students know that the area under the graph of the basic sin function from 0 to $\frac{\pi}{2}$ is one unit, then by symmetry the average value of the *cos* function over the same interval is $\frac{1}{\pi/2} = \frac{2}{\pi}$



2006 - average value of a cos function

Question 8

The average value of the function $y = \cos(x)$ over the interval $[0, \frac{\pi}{2}]$ is

A. $\frac{1}{\pi}$

B. $\frac{\pi}{4}$

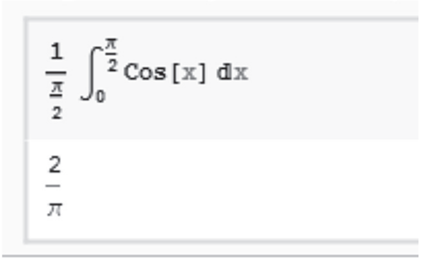
C. 0.5

D. $\frac{2}{\pi}$

E. $\frac{\pi}{2}$

Figure 14. 2006 Examination 2 – multiple-choice Question 8

The majority of students were able to make this connection or correctly formulate as a definite integral and carry out the corresponding computation, as shown in Figure 15:



$$\frac{1}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \mathrm{Cos}[x]\, dx$$

$$\frac{2}{\pi}$$

Figure 15. Evaluating definite integral for average value of a function over an interval

Multiple Choice Item-Response Analysis Task

The notebook format can be used to develop and present an item response analysis task for coursework assessment. This could include:

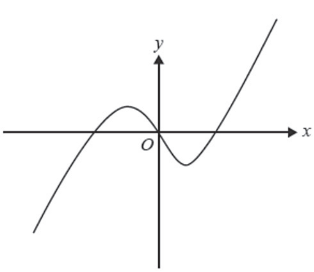
- cut and paste of selected items according to area of study/content focus
- noting the correct alternative and asking students to justify
- asking students to identify conceptual and/or working errors associated with one or more incorrect alternatives
- asking students to respond to a variation on a given item
- asking students to develop a variant item of their own and explain why the incorrect alternatives have been chosen

As an example, a selected topic might be the relationship between the graph of a function and the graphs of its derivative function. From time to time one of the latter multiple choice examination questions is of this type, sometimes given the graph of the original function the students is asked to find the graph of the derivative function, sometimes the other way around. For Question 19 from 2010 the stimulus material is shown in Figure 16:

2010 - determining a possible graph for a function f from the graph of its derivative

Question 19

The graph of the gradient function $y = f'(x)$ is shown below.



Which of the following could represent the graph of the function $f(x)$?

Figure 16. 2010 Examination 2 – multiple-choice Question 19

As well as the graph of an anti-derivative function, the alternatives can include inverse, reciprocal, transformed and derivative graphs of the given function. The student could be given the correct alternative and asked to justify why this is the case by working from the graph of the derivative (what do the zeroes and sign of the derivative mean for a graph of the original function), checking by working from the solution graph (finding it's derivative graph); and checking by attempting to explicitly construct a possible model, as shown in Figure 17:

$$\int (x\,(x^2 - 1))\, dx$$

$$-\frac{x^2}{2} + \frac{x^4}{4}$$

```
Plot[-x^2/2 + x^4/4, {x, -3, 3}, PlotRange → {-2, 5}, Ticks → False]
```

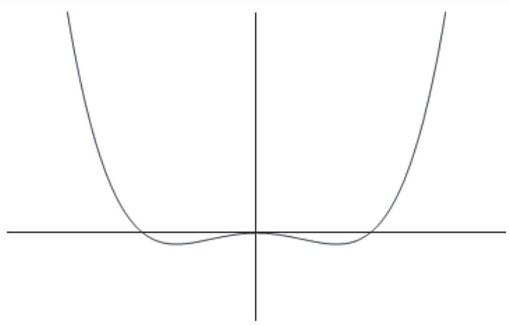


Figure 17. Explicit construction of possible solution

Alternatively, the student could be given the stimulus material and asked to construct four incorrect alternatives, with a comment as to what conceptual or other error each one corresponds.

References

Websites

HREF1: Wolfram Research. (2012). Conference Content Retrieved from http://www.wolfram.com/

HREF2: Victorian Curriculum and Assessment Authority (2012). Conference Content Retrieved from http://www.vcaa.vic.edu.au/Pages/vce/studies/mathematics/cas/casexams.aspx

HREF3: Wolfram Research. (2012). Conference Content Retrieved from http://www.wolfram.com/cdf-player/

USING GEOGEBRA TO ENHANCE TEACHING OF PRIMARY SCHOOL MATHEMATICS

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GeoGebra as an open-source Dynamic Geometry Software has been gaining popularity amongst secondary school mathematics teachers in Singapore. However, the use of GeoGebra in our primary school mathematics teaching has been rather minimal and rare. This paper demonstrates and explains with a few suitable problems on geometry and ratio how GeoGebra constructions and applets can be used to engage pupils in mathematical problem-solving and explorations in an interactive learning platform.

Introduction and Literature Review

The use of information and communication technology (ICT) in the Singapore school curriculum began in a big way with the launch of the first ICT Masterplan (MP1) for education in 1997 and followed by the second ICT Masterplan (MP2) in 2002. By the end of MP2 in 2008, the integration of ICT in the Singapore education system and schools has reached a considerable level of maturity and complexity – each school is equipped with appropriate ICT infrastructure and well-trained technical support staff assisting teachers to implement ICT-based lessons. Today, while we are still in the midst of implementing the third phase of the ICT Masterplan (MP3) we have begun to notice that at least three of the four barriers (accessibility to computers, openness of software, curriculum scope and ICT competence of teachers), identified by Brown (2001), for implementing ICT in the classrooms have been significantly lowered in most schools. Under MP3, there is a demand for developing our teachers' technological pedagogical skills of integrating ICT into everyday classroom teaching with a vision of harnessing the strengths of ICT to transform learners and develop their skills for self-directed and collaborative learning (Ministry of Education, 2008).

In the area of mathematics education, with the innovations of Dynamic Geometry Software (DGS) tools in the early 90s, DGS has become an important platform for teachers to adopt a computer-based and student-centred pedagogy for teaching Geometry. DGS is an open tool that comes with functionalities that are useful for facilitating and enhancing students' learning. In particular, the following two key characteristics are the strengths of DGS tools that would benefit teaching and learning. (a) Interactivity – in a DGS environment, a learner can actively manipulate a mathematical object through "dragging" without affecting the properties of the object. These dynamic and real-time interactions between the software and learners promote curiosity and learning. Indeed, interactive activities that are related to specific learning objectives and designed to match with pupils' cognitive levels can help to engage pupils in their learning (Kalyuga et al., 2003). (b) Visualization – visual reasoning enhanced by the interactivity and the dynamic nature of a DGS tool is especially useful for primary school pupils. The importance of visualization and visual reasoning in students' learning especially at the beginning of a topic has been emphasized by Bishop (1989) and Arcavi (2003) amongst many other researchers. Mason (1992) regards visualizing as "making the unseen visible" which would allow students to see the "abstract" being made concrete which helps to reduce the cognitive burden of learning.

Geometer's Sketchpad (GSP) is one of the leading and earliest DGS tools. It was introduced to Singapore secondary school mathematics teachers in the mid 90s and has since become very popular amongst secondary mathematics teachers. Some local studies showed that GSP was beneficial for students who were learning Geometry (Leong & Lim, 2003; Ho & Leong, 2010). In addition, designing a DGS-based mathematics lesson is considered one of the core ICT skills that pre-service secondary mathematics teachers at the National Institute of Education should accomplish. Indeed, the usefulness and affordances of GSP have helped secondary school teachers in Singapore to alleviate one of the barriers of integrating ICT into mathematics teaching, identified by Brown (2001) as the availability of software which emphasizing learning mathematics.

In the past few years, amongst a few new and powerful ICT tools developed for mathematics teaching and learning, GeoGebra is one of the outstanding and popular ones. GeoGebra, being a free software and having a very large user community offering free online teaching and learning resources (for example from the GeoGebraWiki website), has become one of the most popular and important teaching tools for mathematics teachers around the world. Apart from the standard DGS functionalities, GeoGebra comes with many algebraic and data analysis tools that make multiple representations of a mathematical object easier. One of the important "break-throughs" and strengths of GeoGebra is that its built-in algebraic commands can help teachers to "programme" the constructions. In addition, user-friendly interactive tools (such as sliders, checkbox etc) allow teachers to draw students' attention on the task and at the same time offering flexibilities for students to take control of their own learning.

Anecdotal evidence collected from classroom observations have shown that in Singapore there is an increasing number of secondary mathematics teachers developing resources for GeoGebra and using them for teaching. However, the use of GeoGebra at the primary school level is rare. In fact, the types of ICT tools being used in our primary mathematics classrooms have been confined to using on-line resources or specially-made content-based CD-Roms. We suggest that primary school mathematics teachers should also explore how the strengths and features of GeoGebra could be employed to facilitate pupils' learning and develop them into self-directed learners. Indeed, a study by Yves and Carole (2009) revealed that GeoGebra helps primary school pupils to take on an active role in their learning process.

Examples of Using GeoGebra in Teaching

We should now present a few mathematics problems that were used in our exploratory study on how GeoGebra can be used as a pedagogical tool in supporting teaching and enhancing pupils' learning. The study was conducted in a primary 5 mathematics class (40 pupils) of a school in which the 2nd author is teaching. In this paper, we would not discuss pupils' responses to the GeoGebra-supported worksheet tasks that we designed. Instead we would focus on describing and discussing how these tasks could be used to enhance pupils' learning.

Questions 1 and 2 (Angle Problems)

The problems were modified from one of mathematics papers of the Primary School Leaving Examination (PSLE) - our national examination for all our primary 6 pupils. The geometrical figure shown in Figure 1 was converted to a GeoGebra applet. Since the pupils have not been properly trained to use GeoGebra, we prefer to use the GeoGebra applet as it contains only the essential constructions on the screen. The pupils were guided to run the GeoGebra applet before they answered the question on the worksheet.

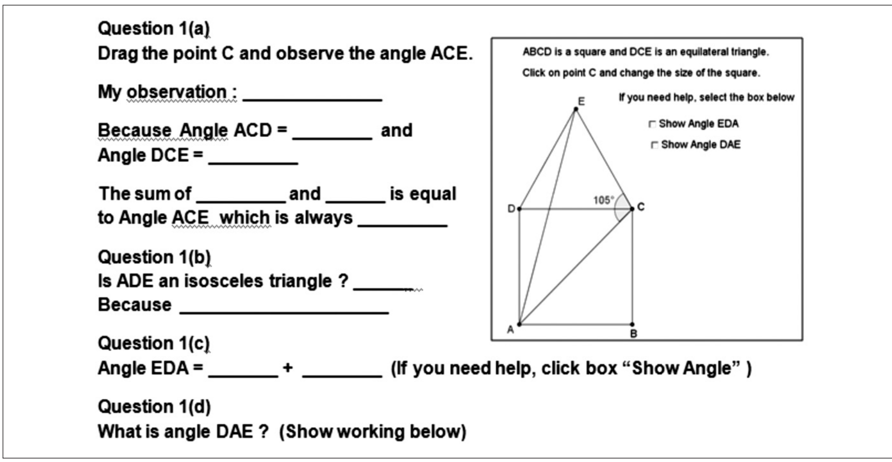


Figure 1. Question 1 in the worksheet.

Figure 1 shows how instructional scaffoldings in the forms of "fill-in-the blanks" and guided steps are used to lead pupils to "uncover" an invariant property - the existence of an isosceles triangle that is inherent in the figure so that they could find the angle DAE. Using GeoGebra as a learning and teaching platform, teachers could now pose problems in an interactive DGS environment. These problem tasks can also be supplemented with appropriate cognitive scaffoldings in the form of instructions or guides which can be shown or hidden on the computer screen via check-boxes. Thus, a traditional and static geometry problem can now be turned into a dynamic one which allows pupils constructing their own knowledge by visualizing and investigating the geometrical object through "dragging". With a little proficiency of using some basic GeoGebra tools or GeoGebra apples, primary school pupils would not be passive listeners in a geometry class anymore as they could carry out independent and self-directed learning if the GeoGebra-supported tasks are well-designed with appropriate on-screen instructions. In addition, to avoid pupils dragging those objects that would probably mess up the constructions and disrupt learning, GeoGebra allows teachers to "Fix an object" (under object property) so that pupils cannot move these objects that are being "fixed".

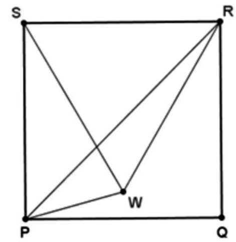
GeoGebra allows teachers to modify or alter an existing construction easily. We constructed a similar diagram but with an equilateral triangle being drawn inside the square in Question 2 (Figure 2). Our intention is to ascertain if pupils could apply what they have learnt in Question 1 (making use of an isosceles triangle) to solve the question without any scaffolding steps. After the pupils have solved the problem, they would be allowed to check

Question 2

Solve a similar problem.

The diagram shows a square PQRS and an equilateral triangle SWR. Find

(i) Angle PSW
(ii) Angle SWP
(iii) Angle PWR



their answers using the pre-constructed GeoGebra applet.

Figure 2. Question 2 in the worksheet.

Question 3 (Application Problem)

Mathematic modeling and applications of mathematics are two main foci of our mathematics curriculum across all grade levels from year 2013 onwards. It is therefore important to let our primary school pupils experience a simple "modeling task" in the form of applying mathematics to solve a real-life problem. The statement of the problem is as follows:

> You are the engineer of a factory. One of the customers want to construct a glass container in the shape of a cuboid with height 4 unit long with stainless steel frames (as shown in the diagram). In order to cut cost (stainless steel is expensive), the requirement of the customer is:
>
> a. the perimeter of the rectangular base must be fixed at 12 units;
> b. the height of the container is 4 units and
> c. the container must have the largest volume.
>
> You should record your investigations in a table shown below.

AB	BC	Volume of the cuboid formed	Total surface area (assuming open top)

This tasked was designed with the use of GeoGebra applet in mind. The pupils were instructed and guided to access the webpage that runs the applet shown in Figure 3.

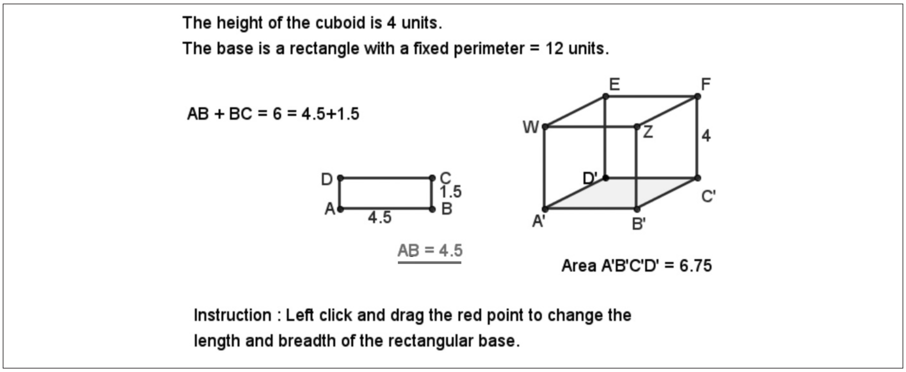


Figure 3. GeoGebra applet for Question 3.

A slider named AB with an incremental value of 0.5 (this is intentional so that pupils would have lesser values to explore) would allow pupils to change the dimensions of the rectangular base. They were reminded to record their observations and perform some calculations in a table provided in the worksheet. Making pupils record and calculate the volume and the total surface area of the cuboid would enable them to experience that even the perimeter of the base is fixed at 12 units, different lengths of AB and BC would still affect the surface area and volume of the cuboid. We hope that this interactive and "hands-on" activity would stimulate pupils' interest in applying mathematics to solve real-life problems. The cuboid on the GeoGebra screen was constructed with an intention to let pupils visualise how the dimensions of the base affect the volume of the cuboid. However, this dynamic 3-D diagram also created confusion for some pupils who asked why the shape of the base is no longer a rectangle even though the area of the base A'B'C'D" is deliberately shown on the screen.

Question 4 (Ratio Problem)

At our primary school level, solving arithmetic word problems using heuristics such as systematic-listing, guess-and-check, working backwards and especially drawing model diagrams is a common and essential classroom activity for engaging students in mathematical problem-solving. In the model-drawing approach, rectangular bars (bar diagrams) are used to represent the quantities and relationships between different quantities in a problem.

Suitably drawn bar diagrams are intended to assist and enable pupils to concretise and visualize the abstract components that are shown or inherent in the problem thereby helping pupils to formulate solution steps. However, drawing bar diagrams may be tedious and time-consuming for some problem situations. The following arithmetic word problem is used to illustrate how teachers can use dynamic and interactive bar diagrams constructed in GeoGebra to assist and enhance their teaching.

> "A sum of money was shared between 3 pupils A, B and C. Pupil A received 1/4 of the total sum of money and the remainder was shared between pupils B and C in the ratio 3 : 2. Find the ratio in which the sum was shared between pupils A, B and C."
>
> As primary school pupils in Singapore do not learn sufficient algebra to solve a word problem like this, a common approach to solving this problem in our mathematics classrooms is by drawing model diagrams as follows:

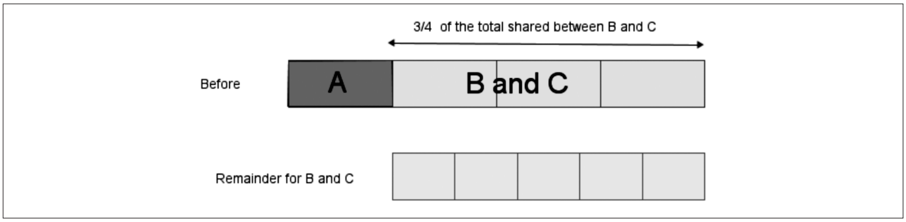
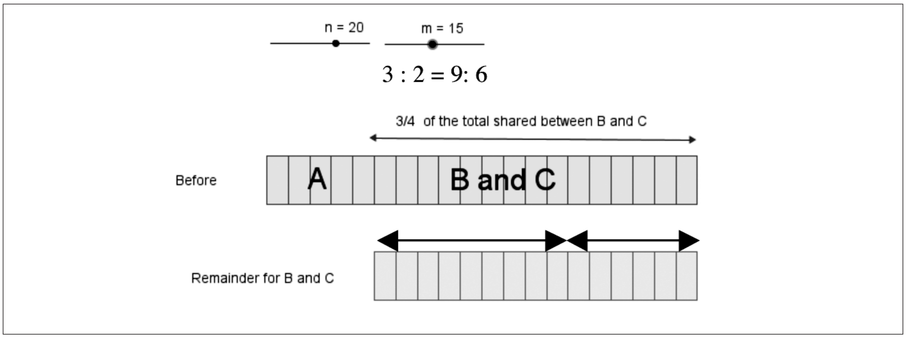


Figure 4. Bar diagrams for Question 4.

Two GeoGebra text input commands, Polygon and Sequence allow teachers to construct a series of rectangles recursively. For example, the first bar diagram in Figure 4 is generated by: Sequence[Polygon[{(k*a /n, 1), ((k + 1)* a / n, 1), ((k + 1)* a / n, 2), (k* a / n, 2)}], k, 0, n - 1] where the value of a is 9 units (a is the total length of the bar diagram and n is the slider value that gives the number of subdivisions of the bar diagram).

In a conventional classroom teaching, teachers would interpret the fractional quantity 1/4 and draw the first bar diagram as shown in Figure 4 followed by explaining that the 2nd bar diagram must be "split" into 5 equal parts in order to obtain the ratio 3 : 2. Obviously, 1 unit quantity of money for pupil A and 1 unit quantity of money for pupils B and C (in the 2nd bar diagram) are not the same – so it is not possible to find the answer from Figure 4. In a non-ICT teaching environment, a teacher would probably have to re-draw and sub-divide the bar diagrams further in order to make comparison again.

With the use of GeoGebra, a new pedagogy would emerge. Teachers would construct



a GeoGebra file beforehand in which two sliders (named m and n in Figure 5) can be used to create different sub-divisions of the original bar diagrams.

Figure 5. Adjusted bar diagrams with more subdivisions.

We should allow pupils to drag the sliders until they create a suitable number of sub-divisions of the bar diagrams so that each rectangle in both bar diagrams align nicely which means that the length of each unit block in the first bar diagram is the same as the length of each unit block of the second bar diagram as shown in Figure 5 above. The action of interacting with the sliders and visualizing how the lengths of the bars change with the slider values would help students to understand the meaning behind creating an equivalent fraction (3/4 = 15/20 in the first bar diagram) and the equivalent ratio (3 : 2 = 9 : 6 in the second bar diagram)–that would be extremely useful for pupils to acquire the conceptual understanding of how this problem is solved.

After explaining how the problem can be solved by using the two bar diagrams, teacher should relate the concept of aligning the blocks nicely (a form of pictorial representation) to an abstract form of representation based on ratios of the quantities. More explicitly, teachers should explain that the total number of units represented by the sum of money for Pupils B and C together in the first model and the second model must be the same (identifying the invariant quantity in the problem) – that is the 3 units in the ratio 1 : 3 = amount for A : amount for B and C and the total 5 units in the ratio 3 : 2 = amount for B : amount for C represent the same amount of money given to pupils B and C together. The concept of equivalent ratios given by 1 : 3 = 5 : 15 and 3 : 2 = 9 : 6 will make the total amount of money received by pupils B and C to be 15 units (9 units for B and 6 units for C). The dynamic nature and interactivity of the Geogebra's constructions of the bar diagrams establish a visual representation of the invariant quantity manifested by the two ratios – which is the key concept involved in solving the problem.

Conclusion

In this paper we have demonstrated how a pupil-centred task (Question 1) can be designed in a GeoGebra environment to enhance the teaching of certain mathematical concepts and techniques. If Question 1 could elicit pupils' understanding of the existence of an isosceles triangle in the figure, it would be helpful for them in solving Question 2. At the primary school level, the main objective of Question 3 is to let students experience how mathematics can be used to solve a real-life problem. Though this maximization problem is more suitable for secondary school students, it can be brought down to the upper primary level with the use of GeoGebra and some simplifications. In fact close to 75% of the pupils of the class obtained the correct answer for the maximum volume. We hope that the process of calculating and recording the total surface area and the volume despite the perimeter of the base being fixed would give these primary school pupils the idea of co-variation between two quantities. Using interactive GeoGebra bar diagrams for Question 4 helps teachers to explain why equivalent ratios have to be used to solve the problem and make pupils understand this concept by visualizing how the alignment of the rectangular blocks in the two bar diagrams is achieved. In conclusion, we feel that using GeoGebra-supported mathematical tasks at the upper primary school level coupled with a pedagogically sound and well-designed worksheet has the potential to enhance teaching and develop pupils' mind for a self-directed and inquiry-based learning - to achieve the visions of our ICT Masterplan 3.

References

Arcavi, A. (2003). *The role of visual representations in the learning of mathematics.* Educational Studies in Mathematics, 52(3), 215–241

Bishop, A. (1989). *Review of research on visualization in mathematics education, Focus on Learning Problems in Mathematics* 11(1), 7–16.

Brown, R. (2001). *Computer Algebra Systems and the challenge of assessment.* The International Journal of Computer Algebra in Mathematics Education, 8(4), 295-308.

Ho, F. H., & Leong,Y. H. (2010, August). *Using dynamic geometry software in teaching geometry proof.* In Y. Shimizu,Y. Sekiguchi, & K. Hino (Eds.), 5th East Asia Regional

Conference on Mathematics Education (EARCOME 5) In Search of Excellence in Mathematics Education, 2 (pp. 520–527).

Kalyuga, S., Ayres, P., Chandler, P., & Sweller, J. (2003). *The expertise reversal effect.* Educational Psychologist, 38(1), 23–31.

Leong,Y. H., & Lim-Teo S. K. (2003). *Effects of Geometer's Sketchpad on spatial ability and achievement in transformation geometry among Secondary Two students in Singapore.* The Mathematics Educator, 7(1), 32–48.

Mason, J. (1992). *Towards a research programme for mental imagery.* Proceedings of the November Conference of BSRLM, (pp. 24–29).

Ministry of Education. (2008). *MOE Launches Third Masterplan for ICT in Education.* Retrieved from http://www.moe.gov.sg/media/press/2008/08/moe-launches-third-masterplan.php

Yves, K., & Carole, D., (2009). *Geobraprim – Geobra for primary school.* Retrieved from http://www.ictmt9.org/files/contributors/5aa0ce1614535cd28043592fafeaa600/ICTMT-9-GeoGebraPrim.pdf.

HIGHER ORDER THINKING TASKS FOR LOW ACHIEVERS IN MATHEMATICS

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Previous studies have found that low achievers make considerable progress when they engage in tasks that foster higher order thinking skills. In this study, three higher-order thinking tasks related to the topic of fractions were assigned to students of mixed ability in Primary 3 (Singapore). We are interested to find out whether these tasks are effective in helping the low achievers understand fractions better and if low achievers can perform higher order thinking tasks as well as high and middle achievers.

Introduction

While teachers tend to use teacher-led whole class instruction to teach low-achieving students and assign traditional drill and practice tasks to them, a number of studies have shown that low-achieving students are capable of exhibiting higher-order mathematical thinking and in fact benefit from activities incorporating higher-order thinking. For instance, Zohar and Dori (2003) analysed four studies and reported that both low-achieving and high-achieving students made considerable progress with respect to their initial scores after they had experienced processes that were designed to foster higher order thinking skills. In one of the four studies, the improvement of the low achievers was significantly higher than for high achievers. In another study, Watson (2001) found that low-attaining learners were capable of making shifts from process to concept. Using a task requiring students to divide identical squares into four equal parts in different ways, she discovered that students could shift from seeing fractions as congruent parts to fractions as equal areas. There were responses on the task that some teachers might have dismissed as incorrect since the learners were known to be low-attaining. However, upon conferencing with the learners, Watson found that the students were cognizant of the concept of equal parts and they were capable of higher-order thinking.

The two studies mentioned involved subjects above the elementary level. In our study, we were interested to find out if the findings were applicable to students in the elementary grades. If so, it would have implications on the kinds of tasks assigned to low-achieving students. It might also mean that their work deserved more examining and one-to-one conferencing might help teachers determine if the students were grasping the concepts taught.

Research Design and Methodology

Subjects

The subjects were 32 Primary 3 students in a mixed ability class. Based on their scores in the mid-year examinations, they were classified into three groups: Low Achievers (below 55 marks), Middle Achievers (between 55 and 85 marks) and High Achievers (above 85 marks). There were 9 students in the Low Achievers group, 13 students in the Middle Achievers group and 10 students in the High Achievers group.

Procedures

Three higher-order thinking tasks were assigned to all the students in the study. The students worked in their ability groups to solve the problems posed. They were given 15 minutes to discuss and write down the solution to each task. There were 2 Low Achievers (LA) groups, 4 Middle Achievers (MA) groups and 2 High Achievers (HA) groups. At the end of each task, 1 LA group, 1 MA group and 1 HA group presented their solutions. After the presentations, the teacher guided the students to see how the answers could be obtained.

In defining higher-order thinking tasks, we referred to Bloom's taxonomy (Bloom et al, 1956). This led us to classify any task that required more than recalling of data or information (classified by Bloom as knowledge) and understanding and interpretation of meaning (classified by Bloom as comprehension) as higher-order. The three higher-order thinking tasks that we assigned to students are as follows:

1. Your teacher will show you a YouTube video clip (Link: http://www.youtube.com/watch?v=wL4hICyMLKU, 1:15 to 1:57). In the clip, two cavemen had 15 bars of gold. The bars were of different sizes (1 whole, 2 halves, 4 quarters and 8 eighths). The cavemen wanted to divide the bars of gold equally among themselves but as the number of bars was an odd number, they could not perform the task. Help the cavemen to distribute the gold bars.
2. ? + ? = ½. Find as many pairs of fractions as you can to fulfill the equation.
3. Write down as many fractions between ½ and ¾ as possible.
4. For each of the higher-order thinking task, fraction strips were provided to the students.

Instruments

The students sat for a pretest before the higher-order thinking tasks were administered. They also sat for a post test after the tasks were completed. We were interested to find out if involvement in the higher-order thinking tasks was effective in increasing the test scores of the students. The pretest and posttest are presented in Appendix 1.

The group presentations were also graded on the process of arriving at the solutions (maximum 2 points) and the accuracy of the solutions (maximum 2 points).

Key Findings

Performance on Tasks

On task 1, all the groups scored zero. The MA group was too shy to present and what they wrote on their presentation sheet was not coherent. The HA and LA groups both said that since there were 15 gold bars, they should divide 15 by 2 and each caveman should get 7 ½ gold bars. They did not consider that the bars were of different lengths.

On task 2, the HA group came up with the least number of solutions, but all were correct. They were also the only group to demonstrate understanding of the concept of equivalent fractions in this task, with responses such as 2⁄8 + 2⁄8 = 4⁄8 = 2⁄4 = ½. The MA group arrived at 7 pairs of fractions, 6 of which were correct. Their last solution was 2⁄7 + 1⁄5. The sum of 2⁄7 and 1⁄5 is very close to ½. Perhaps in their eagerness to beat the other groups, they did not realize that 2⁄7 + 1⁄5 was not exactly equivalent to ½. The LA group started off the task well. They were the only group to start off with pairs of fractions with different denominators (E.g. 1⁄3 + 1⁄6 , ¼ + 2⁄8 etc.). This was interesting as the HA group did not have any pairs of fractions with different denominators and in fact were quite satisfied when they obtained their first answer (2⁄8 + 2⁄8). However, the LA group had several incorrect answers (E.g. 1⁄5 + 1⁄10, 1⁄6 + ¼, ½ + ¼ and 1⁄3 + 1⁄9), showing that they did not entirely comprehend the addition of fractions.

On task 3, the MA and HA groups scored the maximum possible score of 4. The HA group had 9 correct solutions, while the MA group had 4 correct solutions. Both the MA and HA groups demonstrated understanding of how fractions could be compared. The MA group drew accurate pictorial representations of how they had used the fraction strips to arrive at their answers. The LA group wrote equivalent fractions of ½ as their answers, showing that they did not comprehend the task. Their explanation of how they obtained their answers was also incoherent ("We use ½ and compare with the rest to get the answer.").

Table 1

Scores on Higher Order Thinking Tasks

	Task 1	Task 2	Task 3
Low Achievers	0	0	0
Middle Achievers	2	3	4
High achievers	0	4	4

Performance on Pretest and Posttest

Each test had a maximum score of 10 points. Table 2 reflects the average scores of the students in each ability group. The LA group had the most significant improvement while the high achievers' average score on the posttest showed a decrease.

Table 2

Average Scores on Pretest and Posttest

	Low Achievers	Middle Achievers	High achievers
Pre test	2.1	3.7	5.6
Post Test	2.8	4.2	5.4

Implications and Recommendations

Our discussion is focused on the performance of the LA group. The LA group did not seem to understand what Task 1 and 3 required. Their answers lacked coherence. When we planned the study, our intention was to give the students as little assistance and as few hints as possible. We now think that it might benefit the LA students if we gave them more guidance, at least in explaining the requirements of the task.

Furthermore, writing may not be the easiest and most comfortable method for the LA group to explain their solutions. A clinical interview with the LA group after the posttest revealed that they were able to complete Task 3 correctly. Students used the fraction strips to show fractions between ½ and ¾. They were able to articulate that the strips were longer than ½ but shorter than ¾. Noticeably, this did not concur with what was written. Clinical interview offers a richer insight to the understanding of mathematics than written tests. However, it is time consuming and expensive.

For task 1, all students did not seem to have understood the task requirements. Students were given the whole set of fraction strips (it includes fractions with denominators 2 to 12). This might have confused the students. On hindsight, we should have given them only a whole, 2 halves, 4 quarters and 8 eighths. We probably should also have made the instructions clearer and stress the fact that the total length of the gold bars each caveman had was to be the same.

It was also noted that different groups used the given fraction strips differently. For example, some groups were able to compare ½ and ¾ visually without removing any fraction strips from its holder. However, there were groups who removed the strips from the holder and placed the strips side by side to compare the length.

Another observation was that students' performance on the pretest and posttest may have been affected by the test items in the two tests. Although the items in the posttest were similar to those in the pretest, on hindsight, some questions could have been better considered. For example, question 2 required the students to identify the smallest fraction in a set. In the pretest, the fractions in the set were 1⁄3, 5⁄9 and 2⁄7. The smallest fraction was 2⁄7. In the posttest, the fractions in the set were ¼, 4⁄9 and 2⁄7. In this case, the smallest fraction was ¼. Many students wrote 2⁄7, presumably because they recalled the answer from the pretest and did not bother to work out the correct answer in the second test. This posttest item was also slightly more difficult than the one in the pretest. In the pretest, 5⁄9 was clearly greater than ½ and they could straightaway compare 1⁄3 and 2⁄7. In the posttest, all three fractions were smaller than ½, so the item was cognitively more demanding.

The LA group impressed us with the greatest improvement in the posttest scores. Had the learning outcomes for the LA group been simplified, we would have shortchanged their capabilities and potential. This does not imply having a one-size-fits-all curriculum. Instead, schools should not underestimate the abilities of the students and differentiated learning outcome should be carefully crafted.

For future studies, we could consider using a bigger sample size or investigating different topics.

References

Bloom, B., Englehart, M., Furst, E., Hill, W., & Krathwohl, D. (1956). *Taxonomy of educational objectives, Handbook 1: Cognitive domain.* New York: David McKay.

Kaur, B., & Ghani, M. (2012). *Low attainers in primary mathematics.* Singapore, World Scientific Publishing.

Watson, A. (2001). *Low attainers exhibiting higher order mathematical thinking.* Support for Learning, 16(4), 179-183.

Zohar, A., & Dori, Y. J. (2003). *Higher order thinking skills and low-achieving students: Are they mutually exclusive?* Journal of the Learning Sciences, 12(2), 145-181.

Appendix 1

Pre Test

1. What is the missing number in the box?

$$\frac{1}{2} = \frac{\boxed{}}{8}$$

2. Which one of the fractions below is the smallest?

$$\frac{1}{3}, \frac{5}{9}, \frac{2}{7}$$

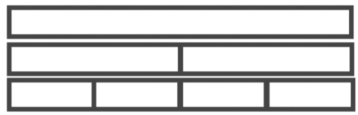
3. Write these fractions in order, beginning with the greatest.

$$\frac{2}{3}, \frac{1}{2}, \frac{4}{9}$$

4. Find the value of $\frac{2}{5} + \frac{3}{10}$. (Express your answer in the simplest form.)

5. Write down a fraction between $\frac{1}{3}$ and $\frac{1}{2}$. Draw a diagram in the space below to show that your answer is correct.

6. Sara and Tricia wanted to share some ribbon equally, but they had difficulty as there were 7 pieces of ribbon. Help them distribute the ribbon such that each of them has the same length of ribbon. Draw how the ribbon can be distributed in the boxes below.



Sara	**Tricia**

7. Write the missing numbers in the boxes.

Post test $$\frac{\square}{4} + \frac{\square}{8} = \frac{5}{8}$$

Post test

1. What is the missing number in the box?

$$\frac{1}{2} = \frac{\square}{8}$$

2. Which one of the fractions below is the smallest?

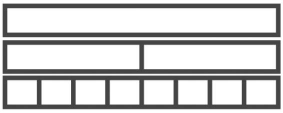
$$\frac{1}{4}, \frac{4}{9}, \frac{2}{7}$$

3. Write these fractions in order, beginning with the greatest.

$$\frac{3}{4}, \frac{5}{8}, \frac{5}{12}$$

4. Find the value of $\frac{1}{4} + \frac{5}{12}$ (Express your answer in the simplest form.)

5. Write down a fraction between $\frac{1}{2}$ and $\frac{2}{3}$. Draw a diagram in the space below to show that your answer is correct.
6. Lisa and Min wanted to share some ribbon equally, but they had difficulty as there were 11 pieces of ribbon. Help them distribute the ribbon such that each of them has the same length of ribbon. Draw how the ribbon can be distributed in the boxes below.



Lisa	**Min**

7. Write the missing numbers in the boxes.

$$\frac{\square}{5} + \frac{\square}{10} = \frac{7}{10}$$

USING AN IPAD IN A MATHEMATICS CLASS

Dennis Fitzgerald

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A number of schools have introduced some form of mobile technology into their classes, either Notebooks or the more recent Apple iPads and then they start to find uses for them in the classes. The iPads give students access to a number of problem solving application. Some are basically open ended applications that can be used in class and allow teachers to add another approach to their range of teaching techniques.

iPads in Education

What are iPads?

iPads are basically small personal computers developed by Apple computers. As with most Apple products, iPads work with Apple approved products only. There are fortunately many software Applications called "apps" that can be downloaded using the supplied iTunes program with many apps being free. They are fortunately quite easy to use and students should be able to use them intuitively if they have already had some computer access beforehand. They use the standard interface that is common to most computers and a touch sensitive screen and there are many stylus type pens as well. It is well worth the money to buy a protective case, a number of which come with a portable keyboard.

The first and most obvious use of these technologies is as a gate to the Internet through Wi-Fi access and a program such as Internet Explorer or Safari. This is also one of the main concerns that a school needs to consider when developing an implementation plan so that Internet access is appropriate for the class program. Students will often go onto Facebook or similar sites when they have the opportunity and they can spend a lot of time online less productively than would be desired. There are a number of suggested implementation plans on education websites.

Text Books

The recently published textbooks from most major suppliers come with digital versions, either on a disk or via website code redemption, normally as PDF files. These files can be easily uploaded to the iPad using the iTunes "add file to library" or "add folder to library". The students then do not need to cart their heavy textbooks around and they will have a textbook with them as they will make sure that they have their iPads at all times.

At recent book publisher presentations, the various major publishers stated that they are working to make their material more iPad friendly and some have produced specific apps that use the code that comes with theirs texts. It is also handy as a teacher to be able to project the textbook questions using interactive whiteboards or data projectors. However, it is worth noting that most of the teacher's editions come with answers.

Teacher Utilities

There are a number of applications that are available for teacher usage and the following are some that have been tested in classes.

Socrative

"Socrative is a smart student response system that empowers teachers to engage their classrooms through a series of educational exercises and games via smart phones, laptops, and tablets." (http://www.socrative.com/). This is an example of "clicker" technology where students give responses to questions, normally multiple choice, that are then collated and displayed automatically when connected to a data projector. This is, perhaps surprisingly, very engaging for the students as they can present their answers anonymously and the class can discuss why some incorrect answers may occur. The discussion of how errors occur can be just as beneficial as going through the correct processes.

This particular software, which is free, is easy to use and does allow a number of question types so that you can have the students do the questions and collect their answers automatically – good for the environmentally concerned of us. It does take a short while to create and upload the tests but they are then reusable and can be shared easily.

Other Interesting Educational Apps

There are a number of websites that are available through iPad apps which can be used for classes and also for teacher professional development.

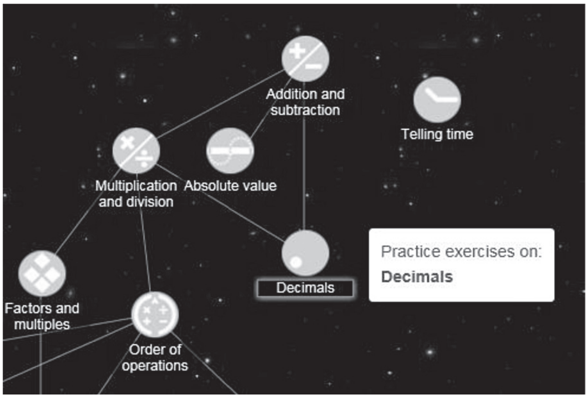
Ted.com

Ted.com is a collection of videos covering a wide range of topics. The app allows you to download and then watch a video at a later time. The videos cover talks from leading experts giving very engaging presentations including topics as broad as "Why is x the unknown?"

(http://www.ted.com/talks/lang/en/terry_moore_why_is_x_the_unknown.html) and there is a very interesting explanation for this. The talks are likely to be advanced for many students although they will also be of interest to teachers.

Khan Academy

The Khan Academy, www.khanacademy.org, is an iPad app that allows access, with an internet connection, to the Khan Academy collection of over 3000 videos that cover a massive number of topics. These videos can be used by students to go over a topic or to review the concepts at a later time. The videos are detailed and free. They are accessible by a search menu or through a graphical interface.



Dropbox

Dropbox is a web 'cloud' application that can be installed on iPads or laptops. It creates a folder that you can invite selected students to 'join' so that they have internet access to this folder and can copy material from it. It is useful for 'dropping' in sets of notes, old tests, solutions, homework materials and it saves a lot of paper as well as photocopy costs. You could drop in a test at the start of a lesson and set it as read only and later add the solutions as well.

Finding Apps for iPads

There are a number of iPad apps available, literally thousands, many of which are free. There are a number of websites that make recommendations relating to educational apps although a simple Google search, such as "educational iPad apps mathematics" will find plenty of examples and they will normally connect straight to iTunes and can be downloaded to the iPad directly. Some starting places for mathematics apps are;

http://appsineducation.blogspot.com.au/p/maths-ipad-apps.html

http://www.mathsinsider.com/16-cool-ipad-math-apps-that-your-child-might-actually-love/

http://mathxtc.com/MacMaths/iPadMaths/iPadMaths.html

Using the iPad Camera

Students often use the camera to record whiteboard notes, either so they don't have to write them down or to pass on to friends. There are also some apps that will record your screen so that you can keep a video copy of any notes you might use when it is used as a data projector. The screen resolution and stylus pen accuracy are not particularly accurate at the moment so it can be difficult to use unless you have good writing skills.

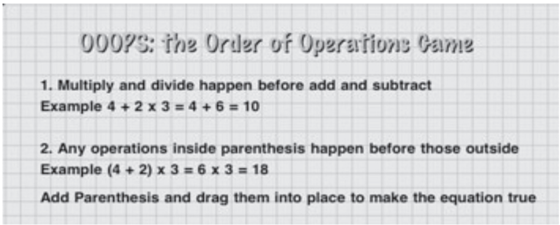
Student Apps

There are thousands of apps available for, some for students, some for mathematics and some are free. Although the cost of most apps is quite low, often $0.99, they can build up. Most apps are described well in their iTunes pages although a few that are 'free' are only free for the beginning level and more levels or chapters can be expensive. On rare occasions some of the apps are not as described and one that showed various steps of calculus operations was actually doing the calculations incorrectly. It is best to search for apps related to a topic such as 'probability' and see what is available. The following apps are some examples related to specific topics and some problem solving examples.

Minds of Mathematics

This is an extensive timeline covering the major discoveries in mathematics with articles and detailed commentaries. However, the app is large and should be downloaded on a fast connection.

Ooops



This is a simple app that asks the students to add brackets at the correct location to make some equations true. (http://itunes.apple.com/us/app/ooops/id467564672?mt=8&affId=1449142) It seems very simple at the start but does become quite complex as it gets to the higher levels.

Learning Programs

There are a number of apps that will solve quadratic equations like Factor. There are games like Bubbles that require students to use their skills with the four operations in order to complete various levels. The old 15 Slider puzzle is also available free.

Puzzles for Students

The available apps include a number of Tangram based puzzles. Other free pattern puzzles are MindPuzzle and Water Pipes. Apart from a number of actual Sudoku problems there are also variations including KenKen which is an extension to Sudoku with mathematical questions – very addictive! Other examples include the 8 Queens problem, the Knight's Tour, mazes, chess games, code puzzles and dice based games such as Yahtzee.

Now go to Apple.com, download iTunes and start exploring!

HOW CAN WE USE GOOGLE EARTH IN MATHEMATICS CLASS

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Google Earth is a free program that most students and teachers are familiar with although most only use it to view their own house. Google Earth has measurement capabilities and works in a number of measurement units as well as longitude and latitude. With the use of GPS devices students can develop a better understanding of shapes and scale.

Google Earth

Google Earth allows users to view the Earth using overlays of satellite photos at various scales with a high level of resolution for capital cities and major cities. The program can be used from the internet with a less detailed version from www.google.com.au although it is best to download the standalone version that will work on laptops and an iPad version is also available. It does require a reasonable speed internet access so that maps can be refreshed quickly.

Mathematical Use of Google Earth: Some Starting Questions

The main tools that mathematics students can use are scale, area and length. From these a number of projects have been developed. Some possible uses are described below. There are a number of questions in the following sections that could be generalised for any location including a local area that is familiar to the students.

Longitude and Latitude

A good starting point is to look at the concept of longitude and latitude by finding values for specific locations. Students can also find specific latitudes such as Melbourne's, 38 South and step around the world in 10° degree intervals and explore the world. They could estimate how many of the points are over oceans and examine their range of errors and repeat this at 30° and 50° south.

Area and Estimation in Sports

By measuring the area of a region such as a park and making an estimate of the area a person might need, such as 1m2 each the students could work out how many people the park could hold and then compare it with official values. When the official seating value is used the actual area per person can be worked out and compared with other values around the world. As part of this the students would work out the area of the Annulus formed by the seating areas surrounding the playing fields.

What is the Playing Area?

Students can investigate answers to questions such as these. Is the MGC the biggest football oval? Do soccer ovals around the world vary in size or surroundings? How much of a sporting venue is a car park, how much is playing grounds and how much is unused?

How Far is the Run?

How far could a marathon take you if the runners started at the MGC and ran in as straight a line as possible? What would be the area covered if they had to 'run around' the bay?

Moving Location

If you found the footy game was at Docklands rather than the MGC how far do you have to travel? What is the actual shortest distance that you would have to travel through the various streets to it?

Perimeter

How far around the various ovals is it? If a running track is set up for a 400m race, what percentage of the oval will be used?

Land Usage

Select several suburbs and a 1km square. Is there a significant difference in the amount of park land, swimming pools, shopping centres or average house block size? Does analysing these values support your judgement of the suburb?

Airplane Landing Strips Around the Country

Who has the longest strips, the greatest total strips, the most strips per overall area, is the strip length related to the car parking area?

Volume

Try to determine the volume of a building by working out its base area and then making an estimation of its height.

What volume of material is stored on the local docks? Find out the volume of a standard container and estimate the number of containers.

Shapes

Locate and determine the perimeter and area of as many different shapes as you can find around the city.

Estimation

Pick a specific item such as cars and estimate the number in a specific location. Zoom in on the object and make a more accurate count.

Data Handling Activities

There are a number of ways of collecting data from different locations that can then be analysed and graphed.

Car Park Count

Count the number of different coloured cars in a shopping centre. Draw a pictogram or bar chart to show your results. How many spare parks are there, in number and percentage?

How Many Can Play?

Select a particular sport such as tennis for a specific suburb and count the courts and multiply by the number of possible players and determine which sports might be the most popular in the area. Present several sports in table and then on a graph.

Food Production

Examine a number of regions and look for quantities of food being grown, number of orange trees, area of vines, areas of wheat production.

Student Estimation of Distance

Have the students pace out a specific distance at school, such as from one end of the school property to the other and check their estimates with Google Earth maps. If you have an iPad with Google Earth loaded then this can be done on site immediately. Does repeating this activity improve their accuracy?

Scale

A scale and height is given on the Google Earth screen. Have the students examine the connection between the height and the scale.

GETTING RID OF THE TEXTBOOK: THEORETICAL BASIS AND PRACTICAL EXPERIENCE

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Textbooks continue to be a very common part of many Australian Mathematics classrooms. This is despite a significant amount of research showing that they are not aligned with current understandings of effective mathematics teaching. This paper reviews this literature and describes the experiences of two teachers who have transitioned away from a reliance on textbooks in their classrooms.

Introduction

Far more than in other subjects, textbooks are ubiquitous in the Mathematics classroom. Research has shown that textbooks continue to significantly influence mathematics classes in Australia and overseas (Pehkonen, 2004; Vincent & Stacey, 2008). There are a number of reasons that schools and teachers may choose to phase out the use of Mathematics textbooks. Foremost amongst these is the growth of one-to-one ICT programs as well as the view of the textbook as a symbol of the traditional, lecture and drill style of teaching mathematics. This paper will look at the roles traditionally played by mathematics textbooks and the experiences of two mathematics teachers moving away from a reliance on the textbook.

Roles Filled by Textbooks

Textbooks play a role in the majority of Australian mathematics classrooms. In the 1999 TIMMS study students used textbooks or worksheets in 91% of the lessons observed (Hiebert, et al, 2003). In the 2002/03 TIMMS study about half of the Year 8 teachers used a textbook as the primary basis for their lesson, with only 5% of classes not having a textbook at all (Thomson & Fleming, 2004).

There have been sustained efforts from professional bodies and researchers for over 30 years to transform mathematics education from the closed, skill and drill, chalk and talk models to the so-called "reform style" with a focus on conceptual thinking and problem solving (Cockcroft, 1982; NCTM, 2000). However, little progress seems to have been made, with the focus still often resting on fluency or basic rote learning. A possible explanation for the lack of movement towards the new style of mathematics education is the perception that many teachers have of the importance of a textbook, with it filling many roles, including:

- ensuring they deliver the specified curriculum (Haggerty & Pepin, 2002; Reys, et. al., 2003)
- assistance in planning and structuring their curriculum (Schmidt, et. al., 2001),
- a guide for pacing and ordering of material within a topic (Freeman & Porter, 1989)
- a source of worked examples and exercises of increasing difficulty (Love & Pimm, 1996).

As many teachers use textbooks to organise lessons and structure their courses, they can dominate what students learn (Apple, 1992). Some teachers supplement their textbook with other materials and alternative activities; however there are still many who do not (Vincent & Stacey, 2008). Further, textbooks usually approach learning mathematics with a 'one-size-fits-all' approach, which can be detrimental to the learning of the majority of students (Siemon, et. al., 2001).

Textbooks offer stability, and are logical and explicit about what is to be taught. When teachers feel overloaded by the demands of teaching, both inside and outside of the classroom, it can be easy to view the textbook as a planning tool, meaning fewer choices to make, with a built in schedule to keep to (Pehkonen, 2004). Many teachers feel passive and mechanistic in their teaching when they use a textbook; however they can feel uncomfortable when choosing to skip over sections, change the order or do only some of the questions. Therefore curriculum materials such as textbooks can be very influential in individual teachers work and can have vast reach within the system (Ball & Cohen, 1996). This can be positive, such as influencing a common curriculum across diverse settings; however it can constrain teaching and limit the opportunities for student learning.

Textbooks, as they are currently used in many Australian mathematics classrooms, are the embodiment of "closed mathematics". Many textbooks organise material into discrete chapters which minimise links between different mathematical concepts. Research suggests that forming links is vital to students retaining concepts and skills beyond the topic test (Stein & Lane, 1996; Vincent & Stacey, 2008). Increasing the intervals between practice sets, from days to weeks or months, has been shown to increase long term retention of mathematical skills (Rohrer & Pashler, 2007), however textbooks encourage a focus on only one skill and concept at a time, very rarely returning to previous skills for later practice.

Mathematics education should encompass all of the different levels of Bloom's taxonomy (Krathwohl, 2002), from the ability to recall and apply basic formulas to the ability to generalise, prove and find solutions to novel problems (Schoenfeld, 2004). In contrast, textbooks traditionally only contain problems of low procedural complexity and do not require students to make connections (Vincent & Stacey, 2008). Therefore textbooks focus mainly on fluency, only one of the four proficiency strands in the Australian Curriculum, the others being understanding, reasoning and problem solving (ACARA, 2009). Further, textbooks are built around individual practice, ignoring the benefits derived from and skills gained during group work.

Experiences in "Getting Rid of the Textbook"

The authors currently teach at two different Colleges in regional and rural Victoria. Warracknabeal Secondary College is a small rural school of 250 students. In 2012 the school introduced a 1-to-1 iPad program at Years 7 and 8 and students were not required to buy a Mathematics textbook. Teachers were given the option to use a pdf textbook, but the author elected not to. Warrnambool College is a school of approximately 1000 students. An optional laptop/netbook program ran for 10 years, and in 2012 the school switched to a full one-to-one netbook program, starting at Year 7, with students no longer purchasing textbooks for the majority of their subjects.

When transitioning away from the use of textbooks, the authors focused on replacing the meaningful roles previously filled by the textbook, as well as implementing a range of open-ended, problem solving and hands-on tasks. The first of the roles filled by the textbook, a source of progressive practice problems, is the most easily solved. The table below presents a number of different ICT based options available.

Table 1.
Alternate Sources of Practice Problems

Name	Positives	Negatives
Khan Academy	Mastery based Hints and videos to support students Great data for teachers and students Encourages spacing Gamified (points and badges) Works on iPads Free	Closed questions Discrete topics Inconsistent difficulty levels Aligned to an American curriculum
Maths Online	Aligned to Australian Curriculum Instructional videos available	Not engaging Medium cost
Manga High	Game based/ engaging Scaffolded question sets Hints and worked examples available Some data available for teachers Ability to set challenges Linked to VELS	High cost Time restrictions on questions No iPad support
Mathletics	Linked to VELS Used in many primary schools	Medium cost No iPad support

Name	Positives	Negatives
Sumdog	Game based – can play against class mates Can set targeted practice areas for individual students Free for basic version	Focuses on number skills only Medium cost for version with teacher data
hatquiz.org	Addresses most curriculum areas Ability to increase and decrease difficulty Free	No in-built progression No help available to students Visually unappealing
Excel Worksheets	Conditional formatting allows for instant feedback Students work at their own pace	Time consuming to create Requires advanced knowledge of Excel
Worksheet Generator	Installed on DEECD computers Easy to use	Not engaging Non-adaptive
iPad apps	Engaging Interactive Varied	Cost and distribution issues Few targeted to secondary Mathematics

Another dominant role of textbooks is as a curriculum checklist. There are a number of alternative sources of curriculum documentation and planning assistance. The Mathematics Association of Victoria has an online resource providing lesson and unit plans, ideas and activities available to members. There are published sources such as Maths in the Inclusive Classroom, two books composed of a series of units aligned to the Australian Curriculum for lower secondary students. These units are organised around a series of practical, themed activities aimed to assist in implementing "just-in-time" teaching (Reilly & Parsons, 2011). Maths300 has a searchable database of investigations and lesson plans accessed via an annual school subscription. The VCAA has published a planning document allowing teachers to cross-check their current curriculum with the Australian Curriculum. The Mathematics Planning Templates can be downloaded from the VCAA website (VCAA, 2012).

Over the year the authors have trialed a range of planning strategies and teaching methods. Although there have been many positives, there has been a tendency to over-rely on resources such as Khan Academy just as it is easy to over-rely on the textbook. A reason for this was the excessive time required to create engaging, integrated unit plans from scratch. Collaborative planning, the use of external resources, experience and the creation of a bank of materials will alleviate this in the future. Resources such as Khan Academy have significant advantages over textbooks in their ability to cater for diverse student needs, the plethora of tracking data provided to the teacher and the instructional videos and assistance available to students. However, they are still a closed form of Mathematics learning and have little emphasis on problem solving and reasoning, as well as a lack of context.

Conclusion

Textbooks currently dominate the Australian mathematics education landscape; however research suggests that an over-reliance on textbooks can have severe consequences on student learning. There are many alternatives which can be used to replace the positive roles currently filled by textbooks, as well as a wealth of resources and assistance to create a more open, investigative classroom. The authors have shown that it is possible to phase out textbooks, even for inexperienced teachers. Although there have been some challenges, it is a vital undertaking which is necessary to improve the standards of teaching and learning in Mathematics.

References

Australian Curriculum and Reporting Authority. (2009). *Shape of the Australian Curriculum: Mathematics.* Sydney, NSW: ACARA.

Apple, M. (1992) *The text and cultural politics.* Educational Researcher, 21(7), 4-11.

Ball, D., & Cohen, D. (1996). Reform by the book: What is – or might be – the role of curriculum materials in teacher learning and instructional reform? Educational Researcher, 25(9), 6-8.

Cockcroft, W. H. (1982). *Mathematics counts: Report of inquiry into the teaching of mathematics in schools.* London: Her Majesty's Stationery Office.

Freeman, D., & Porter, A. (1989). *Do textbooks dictate the content of mathematics instruction in elementary schools?* American Educational Research Journal, 26(3), 259-270.

Haggerty, L., & Pepin, B. (2002). *An investigation of mathematics textbooks and their* *use in English, French and German classrooms: Who gets an opportunity to learn what?* British Educational Research Journal, 28(4), 567-590.

Hiebert, L., Gallimore, R., Garnier, H., Givvin, K., Hollingsworth, H., Jacobs, J., Chui, A., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeck, W., Manaster, C., Gonzales, P., & Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMMS 1999 video study.* Washington, DC: National Centre for Education Statistics, U.S. Department of Education.

Krathwohl, D. (2002). *A revision of Bloom's taxonomy: An overview.* Theory into Practice, 41(4), 2012-2018.

Love, E., & Pimm, D. (1996). *'This is so': A text on texts*. In A. Bishop, K. Clements, C. Keitel, J. Kilpactrick & C. Laborde (Eds.), International handbook of mathematics education (Vol 1, pp. 371-409). Dordrecht: Kluwer.

NCTM. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council for the Teaching of Mathematics.

Pehkonen, L. (2004). *The magic circle of the textbook – an option or obstacle for teacher change.* In M. J. Hoines & A. B. Fuglestad (Eds.), Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 107-136). Bergen: PME.

Rohrer, D., & Pashler, H. (2007). *Increasing retention without increasing study time.* Current Directions in Psychological Science, 16, 183-186.

Reilly, Y., & Parsons, J. (2011). *Delivering differentiation in the fully inclusive, middle years' classroom.* In L. Bragg (Ed.), Maths is multi-dimensional. Proceedings of the 48th Conference of the The Mathematical Association of Victoria. Brunswick, Victoria: MAV.

Reys, R., Reys, B., Lapan, R., Holliday, G., & Wasman, D. (2003). *Assessing the impact of standards-based middle grades mathematics curriculum materials on student achievement.* Journal for Research in Mathematics Education, 34(1), 74-95.

Schoenfeld, A. H. (2004). *The math wars.* Educational Policy, 18(1), 253-286.

Schmidt, W., McKnight, C., Houamg, R., Wiley, D., Cogan, L., et al. (2001). *Why schools matter: A cross-national comparison of curriculum and learning.* San Francisco: Jossey-Bass.

Seimon, D., Virgona, J., Corneille, K., Stephens, M., Griffin, P., Smith, P., Carroll, J., & Callingham, R. (2001). *The middle years numeracy research project: 5-9* (Stage 2), Bundoora, Victoria: RMIT University.

Stein, M., & Lane, S. (1996). *Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project, Educational Research and Evaluation: An International Journal on Theory and Practice*, 2(1), 50-80.

Thomson, S., & Fleming, N. (2004). *Summing it up: Mathematics achievement in Australian schools in TIMMS 2002.* Melbourne: ACER.

Vincent, J., & Stacey, K. (2008). *Do mathematics textbooks cultivate shallow teaching? Applying the TIMMS video study criteria to Australian eighth-grade mathematics textbooks.* Mathematics Education Research Journal, 20(1), 82-107.

VCAA. (2012). *Mathematics planning templates.* Retrieved from http://www.vcaa.vic.edu.au/Pages/foundation10/curriculum/resources/templates/maths.aspx.

THE IMPORTANCE OF NON-VISUAL THINKING WHEN PERFORMING VISUALISATION TASKS

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When trying to investigate how to improve students' ability to transform shapes, I came across the teacher-held view that visualisation couldn't be taught. This paper presents a review of research which suggests that logical deduction is crucial to performing complex visualisation tasks. This process, which uses minimal amounts of visual thinking, can be supported by teaching students to recognise geometric properties and their interconnections.

Overview

Like many Victorian schools, my school dedicates some time to analysing NAPLAN data to look for areas in which we could improve our teaching. When looking across all of our mathematics data since NAPLAN began, my school discovered that not all of our students did well in shape-related questions. In particular, if a question required mentally moving shapes – transformations – the rate of correct responses fell.

Being the leader of the school's Numeracy Improvement Team, I decided to investigate. This paper summarises the information I found in the research literature. I use this information to argue against a view that was prevalent among the teachers that I work with – the view that mentally moving a shape had to be performed by visualising the whole movement.

In the research literature, transformation of shapes falls into the category of spatial visualisation. It's difficult to address any topic connected to geometry without reference to Van Hiele research. Section 2 looks at two modified versions of Van Hiele models and how visualisation fits within the most influential theory used to organise geometry learning. Section 3 examines the differences between Van Hiele theory (which guides research debate) and curricula (which guide teachers).

When visualising shapes transformed, we use several interrelated processes. If not all of the shape to be transformed is visible, then the first step of visualisation is to create the object to be visualised (Gutierrez, 1992). Then, a process that enables us to make the imaginary movement must be applied. Section 4 examines relevant research on the psychological processes of visualisation. This research is about how the object of visualisation is made, the processes of visualisation and the connection between the two.

Van Hiele Research

The Van Hiele Levels are the most influential categorisation of the development of geometric thinking in children. This is not to say that they have not attracted criticism – they seem too hierarchical and too generalised– but they are still seen as very useful by the researchers who make these criticisms (Battista, 2007). Most critics amend, rather than reject the levels outright. Rather than examining all of the amended versions of the levels, we are going to look at just 2 modified versions: Gutierrez's 1992 version, which applies the levels specifically to spatial visualisation and Battista's 2007 version, which adds sub-categories to the levels (both models – which will be discussed in more detail – are presented in Table 1).

While Pierre Van Hiele and Dina Van Hiele-Geldof originally developed the levels to catagorise children's thinking in the development of 2D shape, Gutierrez has attempted to reapply them to spatial visualisation using 3D solids. Battista's (2007) version provides an updated model which applies to geometric thinking in general with subgroups in each level to make the levels less coarse.

While Battista's general model of the Van Hiele levels is based on a large body of research and only differs from the original model in terms of its subgroupings, Gutierrez's model – which has been tested on the analysis of three students' thinking in a few visualisation tasks – is highly theoretical and is the only model to focus specifically on visualisation. As such, Gutierrez's work may be a useful categorisation but it must be remembered that more research is required to determine whether it is an accurate depiction of the development of visualisation.

Table 1
Two Amended Versions of the Van Hiele Levels

Level	Spatial Visualisation Version of Van Hiele Levels (Gutierrez, 1992).	General Version of Van Hiele Levels (Battista, 2007).
1	Recognition: children can only recognise what they can see. They're unable to visualise unseen parts of the shape or movements. They use trial and error to manipulate solids to new positions.	Visual-Holistic Reasoning: students use visual wholes when working with shapes. Orientation has a big effect on recognition. (2 subgroups: pre-recognition and recognition)
2	Analysis: children use global perception and properties of solids (mainly based on observing the solid in question). They can visualise simple movements between concrete positions. The movement is made based on examining the beginning and end position (but not guessed at, like level 1).	Analytic-Componential Reasoning: students can analyse parts of shapes and their relationships. There are three subgroups based on the increasing geometric formality of children's understandings.
3	Informal Deduction: children perform mathematical analysis prior to movement. Some informal justification can be given post movement. They can visualise movements to unseen positions.	Relational-Inferential Property-Based Reasoning: students can now see how properties are related and how some properties are dependent on others. There are 4 subgroups based on the logical sophistication of the relationships made.
4	Formal Deduction: students analyse properties of the object prior to movement and are aware of properties which are not visible (but known through formal definitions). They use pre-planned movements based on mathematical knowledge which are accurate and economical.	Formal Deductive Proof: students can construct formal axiomatic proofs.

Van Hiele research's main contribution to understanding students' geometric thinking in a primary setting is the progression it describes from being able to think about only seen whole objects (level 1) to some knowledge of a shape's properties (level 2) to making connections between known properties (which provides information for unknown properties of the object – level 3). Whether neuroscience has any relevant evidence on the feasibility of this progression will be discussed in section 4.

Van Hiele Theory and VELS – Different Foci

The influence of Van Hiele levels on the shape section of DEECD's Mathematical Developmental Continuum P-10 (Stacey et al., 2006) is immediately apparent. The bulk of primary geometry topics involve moving students from global, visual-holistic (Van Hiele 1) reasoning towards localised, analytical-componential (Van Hiele 2) thinking. Van Hiele theory is less apparent in the learning statements and VELS primary-level standards themselves. This is because VELS standards assess what students can do, not how they do it. Van Hiele theory is more concerned with how students think about geometry. Students with exceptional global perception may be able to 'pass' middle-years levels in VELS while operating in Van Hiele level 1.

The difference between VELS's focus on what children can do as opposed to how they do it highlights an interesting split between the literature and the curriculum around skills such as visualisation. At the practitioner level, this difference is crucial – a classroom which focuses on measuring what students can do (i.e. the level of complexity of the visualisation tasks they can perform) is markedly different to a classroom which measures how students do it (i.e. the complexity of the strategies used to complete the task). Of course, these two approaches are interconnected; proficiency in complex strategies should lead to being able to complete more complex tasks.

I claim that this causes problems for the teaching of visualisation in that many teachers feel that teaching students to develop Gutierrez's level 2 is somehow cheating – that you're only truly able to visualise if you can manipulate the whole object. Decomposing a shape and focusing on parts is some kind of work-around that is not visualisation. The ability to manipulate the whole shape is then, as one colleague put it, "something you either have or you don't; it can't be taught". While claims that these kinds of attitudes are widespread can only be based on anecdotal discussions with colleagues (due to a lack of research), the position that spatial visualisation cannot be taught has, at times, been the prevalent view (Ben-Haim, Lappan, & Houang, 1985). Ben-Haim et al. challenged the view that visualisation cannot be taught by making significant improvements to students' ability to visualise shapes (3D shapes from 2D representations with no movements – creating an object of visualisation with no other processes applied).

Moving from visual-holistic reasoning to analytic-componential reasoning as a means to develop visualisation can be justified by the research. In Gutierrez's (1992) examples, students at higher Van Hiele levels are able to use more technical language to identify properties of a solid which will enable them to visualise the objects transformed (e.g. Carmen, the highest performing student in the study, uses terms such as lateral, degrees, face, edge and perpendicular in contrast to other students' informal language). Similarly, when Battista (1990) tested high school students for gender difference in geometry and analysed which strategies students used (such as visual, analytical or non-spatial strategies), he found that students who did well tended to use more analytical strategies and rely on visualisation less. They knew a plane intersecting 3 points of a 3D shape was going to be a triangle because it intersected three points (their analytical knowledge) and didn't draw or visualise what it looked like. These examples help demonstrate the interconnectedness and dependence of spatial visualisation on general geometric ability and logical deduction. It would be uncommon for children to be able to operate at Gutierrez's Van Hiele level 2 in spatial reasoning without also being able to operate at level 2 in Battista's more general model (i.e. to be able to perform imaginary moves on objects with some localisation without being able to recognise localised properties to begin with). Thus, in order to develop visualisation, recognition and categorisation of the properties of shapes is essential.

The Psychology of Van Hiele Theory and Visualisation

Studies (some by Kosslyn, others by Posner and Raichle) are citied by Battista (2007) to explain some of the psychology of shape recognition. The brain sees shapes globally first, then it can see them locally. Brain damage in one part of the brain can produce problems with global perception, while damage in another area produces problems with local perception. This provides a neurological explanation for "global/local processing separation (being) built into the architecture of the brain" (Battista, 2007). Through learning and development, these separate processes become integrated. As adults, we can engage in both processes without really thinking about it, but children need to develop this integration. At Gutierrez's level 1, children are using global processing only. The progression to higher levels can be seen as the development of local processing and the development of the integration of these two mental processes.

Kosslyn, Posner and Raichle's views on the architecture of the brain raise several questions for educators. Three of these need to be answered before visualisation can be taught: if global/local thinking is used to visualise objects and movements, and it improves partly through education and partly through brain development, then:

1. To what extent can it be influenced by intervention (i.e. to what extent can visualisation be taught)?
2. When should educators intervene to maximise development (i.e. when has the brain suitably developed to be ready for intervention)? And,
3. What kind of intervention works? I shall provide some evidence to answer the first two questions.

The existence of successful interventions in the teaching of visualisation (such as Ben-Haim et al., 1985) implies that it can be influenced by teaching. The extent of that influence is open to debate and requires further research. Kosslyn (1994) argues that, in order to effectively integrate global/local perception, most people will require instruction. This challenges the view put forward by my teacher colleagues; it seems that not only does visualisation require teaching, it requires students to learn how to use strategies that don't seem like visualisation. In the Battista (1990) study mentioned in section 3, two classes were used. Teacher 2 stressed visualisation in her teaching and, subsequently, her female students didn't perform as well in Battista's test as Teacher 1's female students. Battista (2007) uses Kosslyn to claim that children may begin thinking in images but eventually recode images into propositional format or words (Kosslyn himself situates this claim within a long-standing debate within psychology, which may be more problematic than its use by Battista implies). This means that, while I may have to visualise a triangle when first developing the concept, eventually, I can think of a triangle as a collection of related propositions without the image. This mental model can be retrieved as a word with little cognitive load. Processes of visualisation (which may have also been recoded into propositional format) can then be applied more efficiently without the cognitive load of visualising the object to be transformed. Teacher 2, by insisting that students visualise the object may have been forcing students to use Van Hiele level 1 and 2 thinking, when formal deduction would have worked better. This does create a counter-intuitive situation where performing complex visualisation tasks may require a cognitive process with very little visualising.

In the context of mathematics classes, the preferred propositional format used to codify shapes will be part of the discourse of geometry. That is to say, as educators conversant in Van Hiele theory, we would hope primary-aged children's mental model of a triangle is moving from 'a pointy shape' to a shape with three sides and three corners. Given the custom of using geometric objects to teach visualisation, the most effective teaching of spatial visualisation will work within the discourse of geometry. This makes effective geometry learning a prerequisite of learning how to visualise via a logical, deductive process. By helping students learn to recognise near-geometric features in real objects, we can enable them to apply these skills to situations outside of the classroom.

As for when to teach visualisation in a primary setting, Mistretta (2000) aligns Van Hiele level 3 with American junior high school. Thus, primary school geometry can be aligned with movement from visual-holistic (level 1) to analytic-componential (level 2) reasoning. Research into perceptual and imagistic reproduction provides a three-level development of children's thinking which has parallels to spatial visualisation (Clements & Sarama, 2007). When reproducing images, children move from encoding (reproducing what is visible) to reproduction requiring memory to transformation requiring rotation or perspective taking. Rosser, Lane and Mazzeo (as cited by Clements & Sarama 2007) found that pre-primary aged children encode and reproduce from memory, suggesting that the brain development prerequisite to being able to transform shapes has occurred by the time children reach primary school.

Perham (as cited by Clements & Sarama 2007) found that students had the least difficulty performing slides, then flips and then finally rotations on shapes. This can be affected by the direction of the flip or rotation (i.e. a 90° rotation can be easier than a 30° flip). Despite rotations being the hardest transformation to perform, Rosser, Ensing, Gilder and Lane (as cited by Clements & Sarama, 2007) found that 4-5 year olds could perform simple rotations of simple shapes with support. 4-8 year olds got dramatically better at performing rotations. It could be the case that children at this age are performing transformations using only recognition of the object globally (Guteirrez's level 1) but, perhaps, this increase in ability is due to an increased ability to think locally (about the parts of objects), which would suggest that the neurological development of global/local cognition is not a hindrance to intervention by this point.

Conclusions – Spatial Visualisation can be Taught

Perhaps my colleagues were correct: the ability to mentally manipulate a whole, complex shape is something you either have or you don't. The point is that mentally manipulating a whole shape is not the only way to successfully perform spatial visualisation tasks.

Logical deduction provides a way of performing transformations that requires minimal visualisation. This involves:

1. breaking a shape down into parts;
2. performing a transformation on a small part;
3. working out how this change impacts logically on the other parts of the shape; and
4. checking the final result. Without a solid basis in geometry, students are unlikely

to be able to confidently move from step 2 to 3 in this process.

Visual thinking still has an important place in visualisation – particularly when operating in the lower Van Hiele levels – but, as visualisation tasks get more complex, logical certainties known from geometry may become more useful than picturing shapes in your head.

My claim is that many teachers, guided by curriculum standards, rather than the research literature, find this counter-intuitive. They think they have to teach students how to perform visualisation tasks using visual thinking processes, thus they fail in progressing student thinking towards deductive methods. They underestimate the importance of analytical and logical reasoning in visualisation. Evidence to support this claim is limited, however, while there are many studies (like Clements & Samara 2011) which examine teachers under-preparedness to teach geometry in general, there are none which look at this counter-intuitive aspect of visualisation. (Battista, 1990; Clements & Sarama, 2011; Kosslyn, 1994)

References

Battista, M. (2007). *The development of geometric and spatial thinking.* In F. K. J. Lester (Ed.), Second handbook of research on mathematics teaching and learning (Vol. 2, pp. 843-908). Charlotte, NC: Information Age Publishing.

Battista, M. T. (1990). *Spatial visualization and gender differences in high school geometry.* Journal for Research in Mathematics Education, 21(1), 47-60.

Ben-Haim, D., Lappan, G., & Houang, R. T. (1985). *Visualizing rectangular solids made of small cubes: Analyzing and effecting students' performance.* Educational Studies in Mathematics, 16(4), 389-409.

Clements, D., & Sarama, J. (2007). *Early childhood mathematics learning.* In F. K. J. Lester (Ed.), Second handbook of research on mathematics teaching and learning (Vol. 1, pp. 461-556) Charlotte, NC: Information Age Publishing.

Clements, D., & Sarama, J. (2011). *Early childhood teacher education: the case of geometry.* Journal of Mathematics Teacher Education, 14(2), 133-148. doi: 10.1007/s10857-011-9173-0

Gutierrez, A. (1992). *Exploring the links between Van Hiele levels and 3-dimensional geometry.* Structural Geometry, 18, 31-48.

Kosslyn, S. M. (1994). Image and brain. Cambridge, MA: MIT Press.

Stacey, K., Ball, L., Chick, H., Pearn, C., Steinle, V., Sullivan, P., & Lowe, I. (2006). *The Mathematics Developmental Continuum P-10* Retrieved from http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/default.htm

YOU SUNK MY SPACE SHIP!

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When games are used in mathematics classes as a tool to mediate learning, students are able to build new mathematical knowledge, skills and understanding in a context where they feel comfortable, motivated and engaged. Many students would find it ideal to have their mathematics classes to be fun and interesting, including playing games, and the games do not even have to be overly complicated in order to help build number sense and a student's self-concept (confidence and beliefs about mathematics). Games can build visualisation, creativity, and problem-solving skills, and when used collaboratively, can also foster cognitive and metacognitive development, as long as the game does not distract from the goal to learn new mathematics concepts or lose its novelty value. In this paper, suggestions are given to use and modify the game of Battleship to introduce concepts from Directed Numbers, and give students a free platform to develop and consolidate their own understandings of Linear Functions and Graphs, such as generating rules from straight line graphs, and simulating the transformations of rotation and translation by manipulating the gradient and y intercept. Suggestions for further expansion of the game are also given.

Rationale for Games in Mathematics Classes

Mathematics is considered important by most, but on the whole, it is not a popular subject, nor is it considered an easy subject to teach or learn. It was stated by the Review

Panel of the National Numeracy Review Report (Commonwealth of Australia, 2008) that "many Australian students are not learning the basics of mathematics, nor are they being equipped for further study or future employment". Recently, this issue has been addressed by the Australian Curriculum, Assessment and Reporting Authority (ACARA) in the creation of the Australian Curriculum: Mathematics, which aims to ensure that students:

- are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens,
- develop an increasingly sophisticated understanding of mathematical concepts and fluency with processes, and are able to pose and solve problems and reason in Number and Algebra, Measurement and Geometry, and Statistics and Probability, and
- recognise connections between the areas of mathematics and other disciplines and appreciate mathematics as an accessible and enjoyable discipline to study. (ACARA, 2012, p. 3)

It is only through research that the best methods can be found and employed to fulfill these aims.

The Australian Mathematics Curriculum is organised around three content strands (Number and Algebra, Measurement and Geometry, and Statistics and Probability) and four proficiency strands. A student is considered proficient when they can show: Understanding – where students can interpret mathematical information, describe their thinking, make connections between mathematical concepts and are able to adapt and transfer them to unfamiliar situations; Fluency – where students can readily recall definitions, factual knowledge and concepts, use correct mathematical language, can calculate answers and approximations efficiently, and can choose appropriate procedures to find solutions to problems; Problem Solving – where students conduct investigations in unfamiliar situations, and model, make choices, interpret, and formulate to seek solutions, and check validity of solutions; and Reasoning, where students explain their thinking, deduce and justify strategies used and conclusions reached, and can prove truth or falsity of statements.

As a mathematics teacher and tutor, the author is particularly interested in finding ways to improve students' mathematical proficiency, and it has been found that engagement is a significant factor in student achievement. Goos, Stillman and Vale (2007) state that

Students' beliefs about themselves as doers of mathematics, and about particular topics, the nature of mathematics in general, and the mathematics classroom environment contribute to their metacognitive awareness and influence their metacognitive regulation. Self-beliefs also reinforce affects, in particular, attitudinal traits such as motivation, confidence and willingness to take risks. (p. 37 - 38)

It has been suggested that games can be used to improve student motivation and engagement, and give an open-ended platform to explore, build problem solving skills (Amory, 2010), and discover new areas of mathematics (Clarke & Roche, 2010). Therefore, it is possible that if utilised effectively, games could be used in mathematics classes to help students gain the proficiencies stated above and to help achieve the aims of the Australian Mathematics Curriculum. The word "game" has broad usage in the educational literature, and has been used to describe, among other things, environments that simulate real-world situations (Edo, Planas & Badillo, 2009), the linking of problems, puzzles and challenges in a virtual environment (Amory, 2010) and may include the use of Information and Communication Technologies or other media (Main & O'Rourke, 2011). For the purposes of this project, the word "game" will be used to represent any situation where learning occurs through a context of play.

It has been found by Clarke and Roche (2010) that by having their students play a game, key teaching points could be identified by addressing misconceptions that can be found through observation of the students as they play. The game involved having students in pairs rolling dice to give the numerator and denominator of a fraction, and trying to colour in that fraction on a fraction wall. It was found that the students could discover, and grow to understand, many fraction concepts for themselves, such as equivalent fractions, improper fractions and addition of fractions. Students also needed to build visualisation skills (since it is quite a visual challenge to see what combinations of fractions could give what has been rolled), and problem-solving skills from weighing up all possibilities before choosing what combination of fractions is to be coloured. This article highlighted how through playing a game, students generated curiosity and are able to discover mathematical concepts or build mathematical skills for themselves.

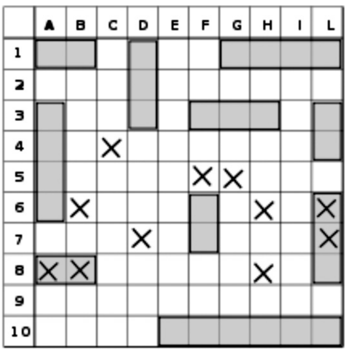
The author would argue that mathematics is the use of numeracy, logical arguments and structured order to make connections and solve problems in unfamiliar situations. Goos, Stillman and Vale (2007) state that in mathematics classes, students need to be given opportunities to appreciate the connection between mathematical ideas and to understand the mathematics behind the problems they are working on in order to gain mathematical proficiency. They believe that an 'inquiry mathematics' approach to learning mathematics can generate higher levels of understanding, and to build metacognitive skills. Games can be used to create this environment of inquiry. Games need to be complex enough to foster cognitive development, visualisation, experimentation and creativity. Amory (2010) makes the argument that there is little change in student performance when the game is merely used as a tutor or transmitter of information, i.e. when the game takes what Amory calls a "learning from" approach, but that knowledge is easily constructed when the game is used as a tool to mediate learning, i.e. when the game takes what Amory calls a "learning with" approach. A collaborative design supports learning, because the discussion of ideas encourages metacognitive strategies such as thinking aloud, recording and modelling, which in turn influence problem-solving abilities.

If you were to ask a group of students their opinions about mathematics, it's likely that the general consensus would be that mathematics is necessary, but complex and unpleasant (Goos, Stillman & Vale, 2007). In research undertaken by Sullivan, Clarke and O'Shae (2010), in which they had students in Years 5 to 8 describe their ideal mathematics lesson, responses were sorted into categories, and the one that had the most responses, with 45% of the 930 responses, was the "fun/interesting" category. Of these, 37% included the use of games. In addition, many students stated that they liked to work in groups and they liked to be challenged, which are important components in games according to Amory (2010) as stated above.

Main and O'Rourke (2011) conducted research to determine whether students' number skills would improve and whether their self-concept (i.e. their beliefs and confidence in mathematics) would improve from playing Dr Kawashima's Brain Training on the Nintendo DS on a regular basis. Essentially this game is skill and drill, but presented in a format that the students find familiar. The researchers observed and recorded the level of engagement and the type of interactions of the students while playing the game, and found that on average the students were engaged for 65% of the time, spent 15% of the time sharing their progress with their teacher or peers, assisting others with operating the game console 10% of the time, and only 10% of the time was spent off task. Interviews were held with students, parents and the teacher, where the students commented that they enjoyed the game and had gained self-concept, the parents commented about the gaining of confidence and the fact that now number skills were "clicking", and the teacher commented about the students' raised level of concentration.

Issues that need to be addressed when incorporating games into the classroom are that the teacher's beliefs about the game to be played, as well as the enforcement of expectations of the class can have a significant impact on student achievement (Main & O'Rourke, 2011). Another issue is that games can be distracting, so it is important to make sure that students stay on task. Finally, games carry novelty value (Main & O'Rourke, 2011), so the researcher needs to ensure, especially if the research is to be conducted over a long period of time that the game does not become stale. It is through the conversations about the mathematics embedded in the games that the learning and language is identified and connected.

Let's Battle!



Battleship is conceptually a very simple game. Played in pairs, each player has a number of ships, which they position on their own game board. Then, in turns, each player says a co-ordinate that they wish to shoot on their opponent's board. The co-ordinates are listed as a letter for the horizontal position, and as a number for the vertical position, e.g. **D3** . The opponent would then say whether one of their ships has been hit or missed.

Figure 1: A traditional game of Battleship.

In order to prevent the game from getting stale quickly, students can be prompted to try to make the game more realistic. The first prompt that could be given is that by using letters to represent a horizontal position, they are immediately restricted to be within 26 units horizontally from the origin. To make it possible to go out as far in either direction as is desired, the only labelling convention that makes sense is to write the co-ordinate as an ordered pair of numbers. This is the perfect opportunity to discuss the idea of a convention – something that has been invented and is now well-known to the public in order to make lives easier and prevent confusion, and it needs to be well-established that the ordered pair always follows the convention that in the ordered pair, the horizontal position is listed before the vertical position. By continually locating and stating co-ordinates to shoot, students can develop fluency in the convention of co-ordinates always being defined by their horizontal position (called '**x**'), then their vertical position (called '**y**').

Direction is Reality

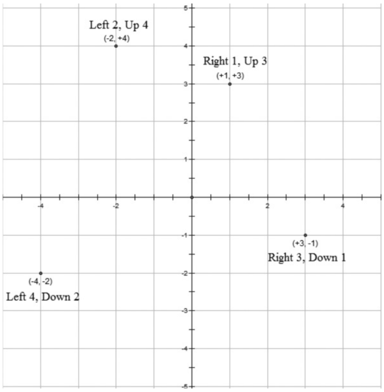


Figure 2: Co-ordinates as Ordered Pairs.

To add realism to the game, the students could be asked questions like "Is it likely that you are going to be backed into a corner, like in this situation we have been playing, and only be attacked from points bounded by the East and the North?" The obvious answer is no, in reality, in war (games), you could theoretically be attacked from any direction. So it is a logical extension to change the field of play from a grid with you backed into a corner (i.e. only playing on one quadrant), to think of yourself as being in a point of *origin* with the ability to shoot in, and be shot from, any direction. Extending the axes to show right and left (East and West) and up and down (North and South) creates the Cartesian plane, and now it is possible to play Battleship over all four quadrants.

This also gives a fantastic teaching opportunity for directed numbers. Your students will realise that in order to state where they want to shoot, they need to not only say how far away the co-ordinate is from the origin horizontally and vertically, *but also in which direction*. Rather than having to say "right, left, up, down" or "East, West, North, South", introduce a new labelling mechanism, where a + is used to denote what seems natural (moving right and up) and a – to denote the opposite (moving left and down). Introducing the idea of a + or – being a direction and the number following representing a distance, and showing these by movement on the axes and grid, will help later when number lines need to be utilised to demonstrate the how four operations (addition, subtraction, multiplication and division) work on directed numbers.

Linear Graphs – Get to the Point

To avoid the game from going stale, and to provide fantastic teaching opportunities when students first learn to draw Linear graphs, is to change the premise of the game to a more futuristic version, set in space, with the aim to destroy all your opponent's spaceships. Students could be asked a question like "If the game was set in the future in space, what sort of weapons do you think they would use?" and those who like science fiction would probably answer "Lasers". Then ask "If you were going to shoot at a co-ordinate with a powerful, destructive laser, would that point be the only point that is hit?" Obviously the answer is no. In fact, every point in the path of that co-ordinate, and every point following the same path past that co-ordinate will also be hit. Graphically, this corresponds to a *line*, and there needs to be some way to tell the onboard computer where to shoot. To do this, we have to use a *rule* that defines a relationship between the **x** and **y** values of all the co-ordinates that lie on this line.

In the modified version of Battleship, the game is set in space, and instead of shooting at single co-ordinates in each turn, each player is in control of a *Death Star*, which shoots powerful, destructive lasers, the trajectories of which each player must control, in order to wipe out the opponent's fleet of spaceships.

Students could be given a point that lies a decent distance away from the origin, e.g. **(4,8)** and asked to draw a line through those two points. In the beginning, it will be best to have the **y** value be an integer multiple of the **x** value, to make finding the rule easier. Have them identify all co-ordinates that have integer **x** value and generate a table of values, and ask if there is an easy way to work out what the corresponding **y** value would be for *any* value of **x**?

Table 1

A Table of Values for $y = 2x$

x	-4	-3	-2	-1	0	1	2	3	4
y	-8	-6	-4	-2	0	2	4	6	8

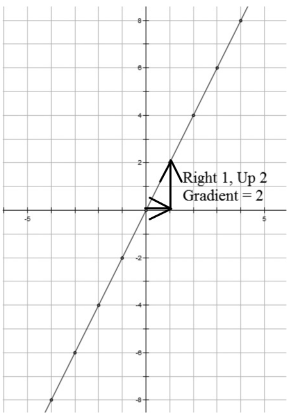


Figure 3: Graph of $y = 2x$

It should be clear that in this case, the **y** value can be determined by multiplying the **x** value by 2. So the rule to get the **y** value from any **x** value would be $y = 2x$. By giving students co-ordinates that have the **y** value as an integer multiple of the **x** value, not only is it easy to generate the rule for each line, but also to see that the number **x** will be multiplied by is the same as the number that the **y** value changes by for each unit change in **x**. This enforces the idea of this multiple of x being a constant rate of increase/decrease, which henceforward can be known as the gradient.

Of course, one would not want to be restricted to only choosing co-ordinates which have each **y** value be a nice integer multiple of the **x** value. Choose another co-ordinate, so that a line can be drawn through it and the origin, where the **y** value cannot easily be read off the graph for each integer **x** value, e.g. $(3,1)$. In this case, only every third value of **x** can be read off the graph.

Table 2

A Table of Values for $y = \frac{1}{3}x$.

x	12	-9	-6	-3	0	3	6	9	12
y	-4	-3	-2	-1	0	1	2	3	4

When it has been established that the gradient is the same as the change in **y** from a unit change in **x**, if this cannot be read from the table of values, students need to make use of some proportional reasoning. From their table, they can see that their change in **y** is not 1, but 3, so is three times as much as is required to evaluate the gradient. To get the unit change that is required, they need to divide it by 3, and so the corresponding change in **y** is also divided by 3, thereby giving a change of $\frac{1}{3}$ in **y** from a unit change in **x**. So the gradient is $\frac{1}{3}$, giving the rule .

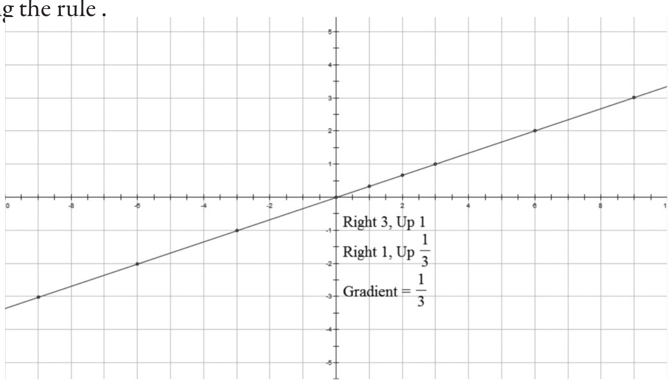


Figure 4: *Graph of* $y = \frac{1}{3}x$.

Following this process, it is easy to generate a method to evaluate the gradient of any line, by dividing the change in **y** by the corresponding change in **x**, which can then be called

$\frac{Rise}{Run}$. What role does the gradient have in the game?

My Head is Spinning

When playing the modified game of Battleship (rules given in Appendix 1), once a player has hit one of their opponent's spaceships at a point, they will need to think of a strategy to hit the ship at other points in order to sink/destroy it. If students have had the chance to simulate what changing the value of the gradient in the rule **y=mx** will have, they will notice that if the value of **m** is positive, the line slopes upwards, and if the value of **m** is negative, the line slopes downwards. They will also notice that increasing the value of **m** makes the line steeper, while a decrease will make the line less steep. Therefore, a change in the value of **m** in **y=mx** corresponds to a rotation about the origin, thereby giving students a strategy to be able to hit a second point on the ship.

As an example, consider the case in Figure 2, where it has been found that the shot $y = \frac{1}{3}x$ has hit a ship at $(3,1)$. It could be guessed that the ship is placed either horizontally or vertically. If the player wanted to shoot at a point one unit above where the ship was hit, a rotation of the laser could be an appropriate strategy. Choosing to have the point (3,2) be shot through, the player would note that from the origin there is a change in **y** of 2 units and a change in **x** of 3 units, thereby making a gradient of $\frac{2}{3}$ and an equation of $y = \frac{2}{3}x$ for the next shot.

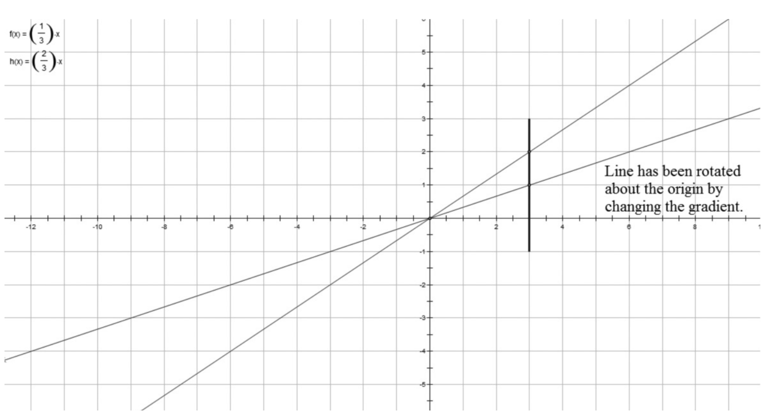
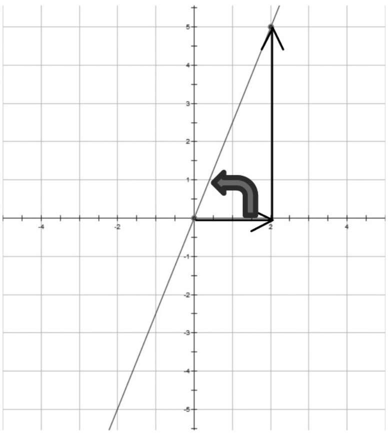


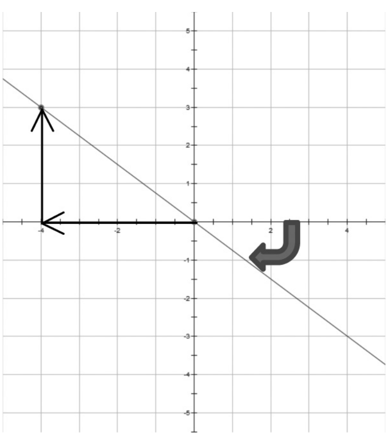
Figure 5. A rotation of the line about the origin by changing the gradient

Evaluating gradients gives a powerful, visual display of some operations on directed numbers, so by gaining some practice with evaluating gradients, and hence, equations, of lines by choosing a point, students may gain a deeper understanding of operations on directed numbers.



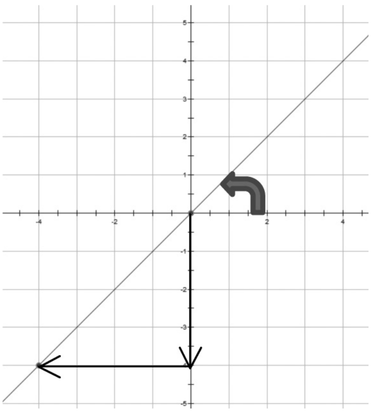
From the origin, choosing a point in the first quadrant, which corresponds to a movement to the right and a movement up, gives a line sloping upwards from left to right, so a positive gradient, thereby showing that $\frac{Positive}{Positive} = Positive$.

Figure 6: Movement Right and Up Gives Positive Gradient



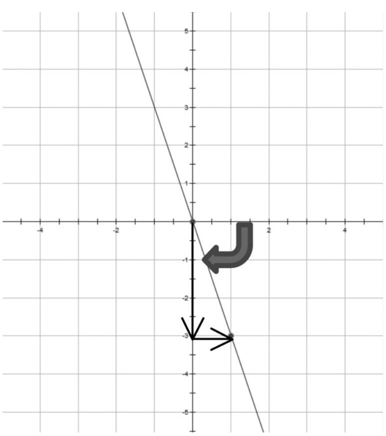
A point in the second quadrant, which corresponds to a movement left and a movement up, will give a line sloping downwards from left to right, so a negative gradient, thereby showing that $\frac{Positive}{Negative} = Negative$.

Figure 7: Movement Left and Up Gives Negative Gradient



A point in the third quadrant, which corresponds to a movement left and a movement down, will give a line sloping upwards from left to right, so a positive gradient, thereby showing that $\frac{Negative}{Negative} = Positive$.

Figure 8: Movement Left and Down Gives Positive Gradient



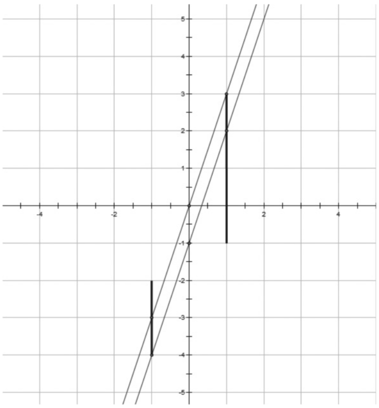
A point in the fourth quadrant, which corresponds to a movement right and a movement down, will give a line sloping downwards from left to right, so a negative gradient, thereby showing that $\frac{Negative}{Positive} = Negative$.

Figure 9: Movement Right and Down Gives Negative Gradient

Lost in Translation

Of course, rotation of a line about the origin is not the only possible strategy that could be employed when playing this game. In the situation shown in Figure 10 below, a rotation about the origin could prove to be detrimental rather than beneficial, due to needing to take two at least two turns in order to hit both ships again. Here, simply moving (translating) the line vertically while keeping the same gradient could enable both ships to be hit at the same time in the next turn.

To introduce the idea of translating a line, give students a simple line, like **y=3x**, and



then ask them to draw a line of the same slope but from one unit down on the **y** axis, so from the point **(0,-1)** . Ask them to generate a table of values from the original line and from the new line. It should be easy for them to see that the values are almost the same, just having been adjusted by having **1** be subtracted from each of the values from the original line **y=3x**. So the equation of the new line must be **y=3x-1**.

Figure 10: Translation of the Line Enables Both Ships to be Hit Again

Table 3
A Table of Values Comparing **y=3x** *to* **y=3x-1**.

x	-4	-3	-2	-1	0	1	2	3	4
y_1	-12	-9	-6	-3	0	3	6	9	12
y_2	-13	-10	-7	-4	-1	2	5	8	11

By experimenting by drawing parallel lines through different intercepts, it should be clear that making a vertical translation is equivalent to making a numerical adjustment to the values of the original graph, and therefore a numerical adjustment to the rule **y=mx** to **y=mx+c**, where **c**, being the number of units adjusted vertically from the origin, also corresponds to the **y** intercept of the graph.

The Sky's the Limit

Since the game is played with the ships positioned vertically or horizontally, it is quite possible that, from the simulation that this game provides, students could discover the rules for vertical and horizontal lines, **x=constant** and **y=constant** respectively, and thereby wipe out entire fleets in just a few swift strokes. To prevent the game from going stale, here are some ideas to further extend the game and make it more realistic.

- With such an emphasis on the idea of direction, it is prudent to discuss other ways that a position could be identified. Instead of "How far horizontally? How far vertically?", why not introduce "How far away in a straight line? In which direction? (i.e. at what angle?)", thereby introducing polar co-ordinates. It is also possible (and advisable) to play on a polar grid, as it better simulates how radar works, and would make it easier to place ships in positions that are neither horizontal nor vertical, while maintaining their correct "length".

- The Death Star could be made more powerful, to not just shoot lasers in a straight line, but lasers that could cover a greater region. To model this phenomenon, linear inequations would need to be employed.
- In space, you are not restricted to be shooting from a single height, but rather a variable height. So why not try playing the game in 3 dimensions?

The benefits of playing games in mathematics are numerous, and the areas of mathematics that are accessible to students from playing and modifying this game are simply mind-boggling, so don't be afraid to experiment. Space is a huge place, so give your students a chance to explore.

Appendix – Rules for Battle Spaceship

These rules have been modified from the rules of Battleship by Hasbro Games (2002). The game can be played with pen and paper, but it is much easier and more beneficial to use a dynamic geometry program. The instructions given are for use with Geometer's Sketchpad, but can easily be adapted to another program or graphing calculator.

Getting Started

- Open Geometer's Sketchpad
- From the *Graph* menu, select *Define Coordinate System*. This will upload a set of axes and a Cartesian grid.
- From the *Graph* menu, select *Snap Points*. This will cause the points you create to "snap" to corners in the grid.
- There are two points already on your grid. The first is at the origin. By clicking and dragging this point, you can move the entire grid. Do not touch this point. The second is at the point (**1,0**) and is used to change the scale of the grid. Click and drag this point so that the vertical axis shows the range [**-5,5**].
- We wish to create a **10x10** grid, so from the Graph menu, select Plot Points. Rectangular should be selected. Plot the points (**5,5**), (**-5,-5**), (**-5,5**) and (**5,-5**).
- Select (highlight) the points (**5,5**) and (**-5,5**) that you have just drawn. From the *Construct* menu, select *Segment*. This will create a segment between the two

points. Follow the same process with the other points to create the outline of your **10x10** grid.

- Highlight the six points on your screen. From the *Display* menu, select *Hide Points*.

Prepare for Battle

You and your opponent may only look at your own screens. You each have in your fleet:

- One *Aircraft Carrier* (takes up 5 coordinates)
- One *Battleship* (takes up 4 coordinates)
- Two *Destroyers* (takes up 3 coordinates)
- One *Patrol Ship* (takes up 2 coordinates)
- One *Death Star* (you do not need to show your Death Star on your grid).

Secretly place your five ships on your grid.

To place each ship, select the *Segment Straightedge Tool* from the left hand side of your screen, and draw line segments to represent your ships. They are to take up as many coordinates as listed above.

- Place each ship in any horizontal or vertical position within your grid, including on or over the axes, but not diagonally.
- Do not place a ship so that any part of it touches or overlaps the edge of the grid or another ship.
- Do not change the position of any ship once the game has begun. Your Death Star is the only part of your arsenal that may move during battle.

Your *Death Star* is equipped with extremely powerful lasers that can shoot in a straight line in both directions and cause damage to anything in its path. You need to tell the onboard computer of the *Death Star* where to shoot the lasers, by inputting the equation of trajectory.

How to Play

Decide who will go first. You and your opponent will alternate turns, calling out one equation of your laser's trajectory per turn, to try to hit each other's ships.

Call Your Shot

On your turn, pick a target position on your grid and call out the equation of a line which will pass through this point.

To show this line on your grid, select *Plot New Function* from the *Graph* menu.

In the *Equation* drop-down menu, select **y=notation**, then type in the Right Hand

Side of the equation of your graph. E.g. if you wanted to graph **y=2x** , you would type **2x.** Select *OK* and your laser's trajectory will be shown on your grid.

When you call a shot, your opponent must tell you whether your shot has hit anything or has missed.

It's a Hit!

If your laser has hit any ships on your opponent's grid, your shot is a hit! Your opponent tells you which ship(s) you have hit (*destroyer, aircraft carrier*, etc.) and the points at which they have hit.

Record your hits by plotting the point. This is done through *Plot Points* in the *Graph* menu. Have all the points that represent something that you have hit in red.

It's a Miss!

If your laser does not hit anything in your opponent's fleet, it's a miss. Keep the graph of the trajectory on your grid though, to show what you have already done.

Your Opponent's Turn

Your opponent will follow the same procedure as you did. You will graph their laser beam in the same way you graph your own, but use a different colour. You can change its colour by selecting (highlighting) the line, and choosing Color from the *Display* menu.

If the laser beam hits any of your ships, click on the intersection of the "laser beam" with your "ship" by moving your curser over the intersection and clicking. Change its colour in the same way that you changed the colour of the line.

Destroying Ships

Once a ship has as many hits as its length (e.g. an *Aircraft Carrier* has been hit in 5 different spots, a *Destroyer* has been hit in 3 different spots), it has been destroyed. The owner of the ship must announce which ship was destroyed.

Winning the Game

The first player to destroy your opponent's fleet of five ships wins the game.

References

Australian Curriculum, Assessment and Reporting Authority. (2012), *The Australian Curriculum – Mathematics.* Retrieved from http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10.

Amory, A. (2010). *Learning to play games or playing games to learn? A health education* *case study with Soweto teenagers.* Australian Journal of Educational Technology, 26(6), 810–826.

Clarke, D., & Roche, A. (2010). *The power of a single game to address a range of important ideas in fraction learning.* Australian Primary Mathematics Classrooms, 15(3), 18–24.

Commonwealth of Australia (2008). National numeracy review report. Commissioned by the Human capital working group, Council of Australian Governments.

Goos, M., Stillman, G., & Vale, C. (2007). *Teaching secondary school mathematics.* Crows Nest, N.S.W: Allen and Unwin.

Hasbro Games. (2002). Rules of battleship. Pawtucket, RI, USA.

Main, S., & O'Rourke, J. (2011). *New directions for traditional lessons: Can handheld game consoles enhance mental mathematics skills?* Australian Journal of Teacher Education, 36(2), 43–52.

Sullivan, P., Clarke, D., & O'Shea, H. (2010). *Students' opinions about characteristics of their desired mathematics lessons.* In L. Sparrow, B. Kissane & C. Hurst (Eds.), Shaping the future of mathematics education. Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia (pp. 531–538). Fremantle

MATHEMATICAL PEDAGOGY – TRADITIONS, TECHNOLOGY AND LUMERACY

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Teachers around Australia have made significant progress in recent years towards achieving effective integration of technology in their classrooms, in the senior and junior secondary years, to enhance and engage students' learning of mathematics. Computer Algebra System (CAS) similar to Texas Instrument's TI-Nspire machines when combined with classroom networking i.e. connected classrooms, similar to Texas Instrument's 'Navigator' wireless communication systems enhance the pedagogy and 'risk taking' by the learners. These combined with the use of Lumeracy resources will engage and inspire the students. Teachers have also been using other CAS machines, however without the facility for connected classrooms.

The word Lumeracy was coined by Kuppuswamy Ramakrishnan (known as Rama), Teacher and Educationalist in 2011 who defined it as a word to represent being educated with knowledge to read, write and use numeracy, manage information, express ideas and opinions, communicate in an ethical manner, make decisions and solve problems.

Aspects of Mathematical Pedagogy

Mathematical pedagogy has been undergoing a paradigm shift albeit slowly since the 1950's due to the beginnings of space exploration requiring the learning of functions as well as patterns, higher algebra rather than basic arithmetic and geometry as well as vocational mathematics. Prior to this teaching was based on learning facts and completing repetitive sums rather than conceptual and contextual learning. Although this shift meant drastic revisions to teaching from the past, the catalyst was needed for people to acquire skills required in computing, functions and advanced spatial concepts and the teaching of algebraic concepts. That was the beginning when one could say that "(m)athematics education has broke[n] free of its chains" (Bergamini, 1970).

To meet such complex needs, teachers around Australia have made significant progress in recent years towards achieving such change by the effective integration of technology in their classrooms, in the senior and junior secondary years. This has enhanced students' learning and engaged them in mathematics. Australian teachers have been at the forefront of the development of both innovative and powerful curricular resources and effective pedagogic strategies from which teachers elsewhere have been able to benefit. These skills, strategies and resources have led to the use of the next generation of mathematics and science learning tools which are embedded in Computer Algebra Systems (CAS).

According to the Cockcroft Report, 1982 (as cited in Grimison & Pegg, 1995), mathematics teaching at all levels should include opportunities for exposition by the teacher, discussion between teacher and pupils and between pupils themselves; appropriate practical work, consolidation and practice of fundamental skills and routines, problem solving, teaching the application of mathematics to everyday situations, and investigational work.

In addition, the National Statement on Mathematics for Australian Schools (AEC, 1991) had a lot to say about the importance of language and culture in mathematics teaching and learning. Mathematics was no longer seen as a body of collected facts, routines and skills to be passed on to students but a dynamic creative process involving invention, intuition and discovery.

However, "a great deal of energy continues to be expended in educational debates on the nature and significance of literacy and numeracy" (Chapman & Alison, 1990). There is an increasing demand on language skills as students talk with each other and their teacher about their mathematical activities. In the process they begin to construct mathematical concepts for themselves. Further, mathematics today needs to be connected to other fields of study. The *Australian Curriculum: Mathematics* (ACARA, 2012) states:

> *mathematics is composed of multiple but interrelated and interdependent concepts and systems which students apply beyond the mathematics classroom. In science, for example, understanding sources of error and their impact on the confidence of conclusions is vital, as is the use of mathematical models in other disciplines. In geography, interpretation of data underpins the study of human populations and their physical environments; in history, students need to be able to imagine timelines and time frames to reconcile related events; and in English, deriving quantitative and spatial information is an important aspect of making meaning of texts. (ACARA, 2012)*

Current State of Play

Technology in the form of Computer Algebra Systems (CAS) and/or CAS along with the Texas Instrument's *Navigator* wireless communication systems goes hand in hand with mathematical pedagogy. It provides multiple representations of functions, dynamic geometry and sophisticated statistical analysis in addition to routine mathematical computations. For those who are unfamiliar with the 'navigator' system, the Texas Instrument (TI)'s Computer Algebra System (CAS) machine which is called the TI-Nspire, communicates wirelessly with the teacher and students through a wireless desktop unit that displays all CAS screens and information on to the whiteboard through software installed on the computer via a data projector. Setting up of a class takes couple of minutes. A class list with names is typed into a new list and students assign their own passwords. When you start the class by clicking on the "start the class" command, students get a login screen and they can communicate by typing in information. The teacher can send files to students and can receive answers in different available formats. The teacher may choose to run the class with all students' screens on the whiteboard. In addition the teacher can make one student the presenter (showing only that student's screen in real time).

Lumeracy

The issue of language is best addressed by the concept of *Lumeracy*. I propose the coining of a new term *Lumeracy* and defining it as "a word to represent being educated with knowledge to read, write and use numeracy, manage information, express ideas and opinions, communicate in an ethical manner, make decisions, and solve problems".

There is also still a great deal to learn about effective ways of using technology in the teaching of mathematics. While technology can enhance and /or show multiple representations, improve the mathematical pedagogy, language becomes critical in clarifying and interpreting problems and the results obtained using mathematical methodology. This obstacle could be overcome by making students lumerate, as opposed to literate and numerate in the traditional sense.

What is considered as a Lumeracy resource is one that is rich in thinking in mathematical ways and crosses all learning areas of knowledge. It connects natural world with its mathematical beauty, the history of mathematics along with the mathematicians and the culture of their time with current world connections. Lumeracy resources make the learning of Mathematics interesting, absorbing, enjoyable and above all fun.

Some examples of the resources, most of which can be obtained online are:

- Mathematics – a human endeavour (Jacobs, 1992);
- Mathematics- a time-life book (Bergamini,1970);
- The man who counted (Tahan, 1993);
- Math through the ages, a gentle history for teachers and others (Berlinghoff & Gouvea, 2002);
- Anno's mysterious multiplying jar (Anno & Anno, 1983);
- The number devil: A mathematical adventure (Enzensberger, 1998);
- Fractals, googols, and other mathematical tales (Pappas, 2010);
- What's your angle, Pythagoras? A math adventure (Ellis, 2004);
- The adventures of Penrose the mathematical cat (Pappas, 1997);
- Mathematics appreciation (Pappas, 1993);
- The magic of mathematics: Discovering the spell of mathematics (Pappas, 1994);
- The music of reason: Experience the beauty of mathematics through quotations. Pappas,1995);
- Mathematical scandals (Pappas, 1997); and
- Math-a-day: A book of days for your mathematical year (Pappas, 1999).

Conclusion

Several schools around Australia are using Computer Algebra Systems (CAS) and/or CAS along with wireless communication systems to enhance the pedagogy and 'risk taking' by the learners. CAS has facilities for calculations, graphing, spread sheeting, statistical analysis, dynamic geometry, document organising and storing and commands and the system is compatible with a PC to download and upload files.

Anecdotal evidence suggests and supports the research that this kind of teaching and learning provides for deeper opportunities to learn concepts and the ingredients for student success. Networking capabilities also increases students' participation, engaging students in mathematical thinking and anonymous communication and the opportunity for open discussions makes thinking 'visible'. This system fits in well with what has been learned in the past on how students learn. Learning occurs when a student makes sense of a new concept/idea and takes 'risks' and learning necessarily involves interactions with other learners (Perso, 2000)

Further research is needed to fill the gaps in the existing literature and to help curriculum developers with cues on use of technology and the type of technology. While research work has been undertaken in Australia on the use of technology in classrooms very little research work has been carried out in the use of wireless technology for connected classrooms. Some research work has been undertaken in United States of America. The results of a pilot study at a private school, which has used the wireless classroom networking system for a few years, (Ramakrishnan, 2011), showed that students and teachers are positively in support of the use of CAS in combination with the wireless system and they have stated that this combination has improved the conceptual understanding to some extent. Conceptual understanding being a cornerstone in mathematics learning and leads to general heuristic thinking which in turn helps students in solving mathematical problems. These problem solving skills are essential in further higher mathematical studies and in the successful mathematical functioning of an individual. Connected classrooms with the facility for student anonymity make them 'risk takers' with consequent improvement in their learning. The power of instant diagnostic capabilities gives a superior edge to the teacher's skills.

References

ACARA (2012) Retrieved from http://www.australiancurriculum.edu.au/Mathematics/Rationale.

Anno, M., & Anno, M. (1983. *Anno's mysterious multiplying jar.* New York: Penguin Putnam books.

Australian Education Council. (1991). *A national statement on mathematics for Australian schools,* Carlton, Vic: Curriculum Corporation of Australia.

Barry, K., & King, L. (1999). *Beginning teaching and beyond.* Third Edition, Social Science Press.

Bergamini, D. (1970). *Mathematics.* Nederland, N.V: Time-Life International.

Biggs, J., & Moore, P. (1993). *The process of learning.* Third Edition, Prentice Hall.

Centre for Technology in Learning. (2006). *Handhelds: What research says and what* *educators can do.* Menlo Park, CA: Prepared for Texas Instruments TI-Nspire Math and Science Learning, Author.

Chaff, W. L. (1985). Linguistic differences produced by differences between speaking and writing. In D. Olson (Ed.), *Literacy, language and learning – the nature and consequences of reading and writing,* pp. 105-123.

Chapman, A., & Lee, A. (1990). Rethinking literacy and numeracy. *Australian Journal of Education,* 34(3) 277-289.

Wilson, C., & van Winsen, E. (2009). *Case Study #32- exploring quadratic functions with TI-Nspire Navigator.* Texas Instruments.

Dawe, L. (1995). Language and culture in the teaching and learning of mathematics. In L. Grimison & J. Pegg (Eds.), *Teaching secondary school mathematics – theory and practice.* Harcourt and Brace. pp. 1-14.

EDWA. (1998). *The curriculum framework.* Curriculum Council, WA.

EDWA. (1998). *Mathematics outcomes.* Curriculum Council, WA.

Ellis, J. (2004). *What's your angle, Pythagoras? A math adventure.* MA: Charlesbridge Publishing.

Enzensberger, H .M. (1998). *The number devil: A mathematical adventure.* Metropolitan Books.

Good, T., & Brophy, J. (1997). *Looking in classrooms.* Fourth edition, Harper and Row.

Grimison, L. (1995). What is mathematics and why teach it? In L. Grimison & J. Pegg (Eds.), *Teaching secondary mathematics – theory into practice,* Harcourt and Brace.

Jacobs, H. (1992). *Mathematics – A human endeavour.* 3rd Edition, New York: W. H Freeman and Company.

Hodge, B. (1990). *Teaching as communication.* Longman.

McInerney, D., & McInerney, V. (1998). *Educational psychology: Constructing learning,* Prentice Hall.

Marsh, C. (1996). *Handbook for beginning teachers.* Malaysia, Longman.

Mercer, C., & Mercer, A. (1996). *Teaching students with learning problems.* Macmillan Publishing Company.

Milton, K. (1995). The mathematical experience. In L. Grimison & J. Pegg (Eds.), *Teaching secondary mathematics – theory into practice.* Harcourt and Brace.

Olson, J., Im, S., Slovin, H., Olson, M., Gilbert, M., Yin, Y., Brandon, P. (2010). *Effects of two different models of professional development on students' understanding of algebraic concepts.* Paper presented at the annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, The Ohio State University, Columbus, OH. Retrieved from http://www.allacademic.com/meta/p429338_index.htm.

Pappas, T. (1993). *Mathematics appreciation.* Tetra, CA: Wideworld Publishing.

Pappas, T. (1994). *The magic of mathematics: Discovering the spell of mathematics.* Tetra, CA: Wideworld Publishing.

Pappas, T. (1995). *The music of reason: Experience the beauty of mathematics through quotations.* Tetra, CA: Wideworld Publishing.

Pappas, T. (1997). *The adventures of Penrose the mathematical cat.* Tetra, CA: Wideworld Publishing.

Pappas, T, (1997). *Mathematical scandals.* Tetra, CA: Wideworld Publishing.

Pappas, T (1999). *Math-a-day: A book of days for your mathematical year.* Tetra, CA: Wideworld Publishing.

Pappas, T. (2010). *Fractals, googols, and other mathematical tales.* Tetra, CA: Wideworld Publishing.

Perso, T. (2001). *Outcomes focused mathematics education.* Perth: MAWA.

Ramakrishnan, K. (2011). *Mathematical pedagogy - tradition, technology and teaching: Towards a new practice.* Paper presented at the AAMT-MERGA Conference, Alice Springs, July 2011.

Tahan, M. (1993). *The Man who counted,* W.W Norton & Company Ltd., London).

REVITALISING AN OLD FAVOURITE: USING INTERACTIVE NSPIRE CAS TECHNOLOGY TO TRANSFORM A GOOD LESSON INTO A GREAT LESSON

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Experienced teachers will know particular lessons they've taught in the past which have really hit the mark. No matter when they were first developed, many of these good lessons can be improved through judicious use of interactive mathematics technology. The author uses three examples of recent advances in Nspire CAS technology to rewrite a 2009 lesson plan in a manner which is designed to increased student participation. Teachers are then challenged to consider doing the same to their own favourite lessons.

The Old Lesson

In 2009, researchers from the Melbourne Graduate School of Education's *New Technologies for Teaching Mathematics* project developed a lesson known as the Surd Spiral for their research schools, based on earlier work by Stacey and Price (2005). It used TI-Nspire CAS technology to explore patterns in surd expressions through a geometric model (HREF1, 2009). The lesson materials which were developed at the Victorian Year 10 level used the various features of TI-Nspire CAS which were available at the time, including the ability of the Lists & Spreadsheets application to record values in exact (in this case, surd) form in a spreadsheet. This supported the goals of the lesson, which revolved around students discovering the usefulness of exact/surd expressions in being able to predict and generalise patterns (HREF2, 2009).

Documentation developed by the researchers included a detailed lesson plan with technology hints for the teacher, and a student worksheet. The style used in the worksheet was one where some general and some specific instructions were given to students regarding their exploratory activity. The original structure of the lesson was based on a series of teacher-led whole-class acivities. The students would be directed to read some information, perform mathematical tasks, discuss and compare interim answers within a small group, and write generalisations and intermediate conclusions at regular intervals throughout the lesson.

As with most mathematics lessons, its success could be measured by the extent to which students contributed verbal as well as written responses to questions posed by both static worksheet and engaging teacher. Students working on an in-class investigation in small groups however, are often prone to letting the clever one amongst them act as spokesperson for the group. Typically the process work of the majority of students is often not apparent within in this type of lesson.

New Possibilities

Recent developments in technology (here, TI-Nspire CAS Navigator software) now allow teachers to constantly monitor individual students' handheld devices in real time, conducting timely formative assessment designed to gauge the extent of understanding and engagement within the entire class. Digital images can be analysed on handheld devices using geometric and graphical applications. Students can also be empowered to share their findings and solution techniques using a student-as-presenter feature, providing avenues for expression and engagement for students less inclined to "come up and show us how you got that". Expanding the number of student contibutors to discussion can positively change the culture of the classroom to one in which making mistakes provides a necessary pathway to new learning opportunities. This would be consistent with the teacher discourse patterns named by Turner and Patrick (2004) as being critical to improving learning outcomes for all students.

Examples of how these techniques can be used to enhance the Surd Spiral lesson follow. It is recommended that the reader familiarise her/himself with the student worksheet found on the RITEMATHS site (HREF1) before continuing.

Working with Digital Images

The Surd Spiral is created by building a series of succesive adjacent right-angled triangles around a fixed point (in Figure 1 below, point A) and starting with the familiar $1, 1, \sqrt{2}$ isosceles triangle (ΔABC or Δ1, below) so that for any triangle other than the smallest the shortest side is 1 unit in length. Also, the middle-length side is the hypotenuse of its preceding triangle.

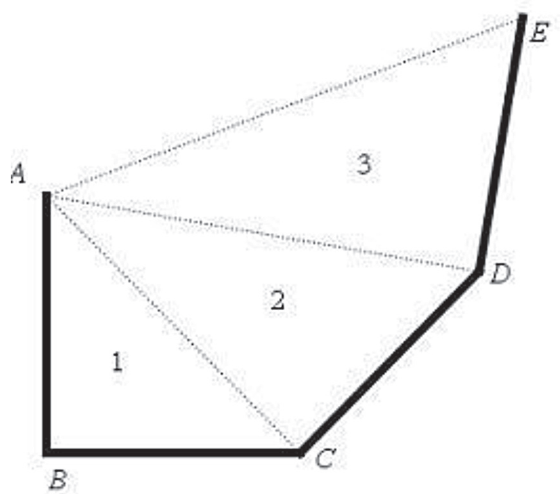

Figure 1. The first three triangles of the Surd Spiral

The diagram shown in Figure 1 is seen in the student worksheet in Activity 1 on page 1. In the original version of the lesson, students use this diagram to place the next point (F) in its approximate position. Teachers in a traditional classroom would walk around and observe random students placing F (using a pencil and a ruler) one unit from E and on a perpendicular to $\overline{AE}$; students in a small group might compare their placements by swapping worksheets. The teacher's interaction would be with one or two small groups of students, or perhaps individuals. Time constraints would prevent all students' work from being observed in the interests of moving on with the lesson.

In an Nspire Navigator classroom, the image (saved as a .jpg file) seen in Figure 1 can be inserted as the background to a Graphs or Geometry application within a one-page Nspire document sent to the students' handhelds at the moment the teacher decides the class is ready to work on it. Figure 2 shows how this would appear in a Graphs application.

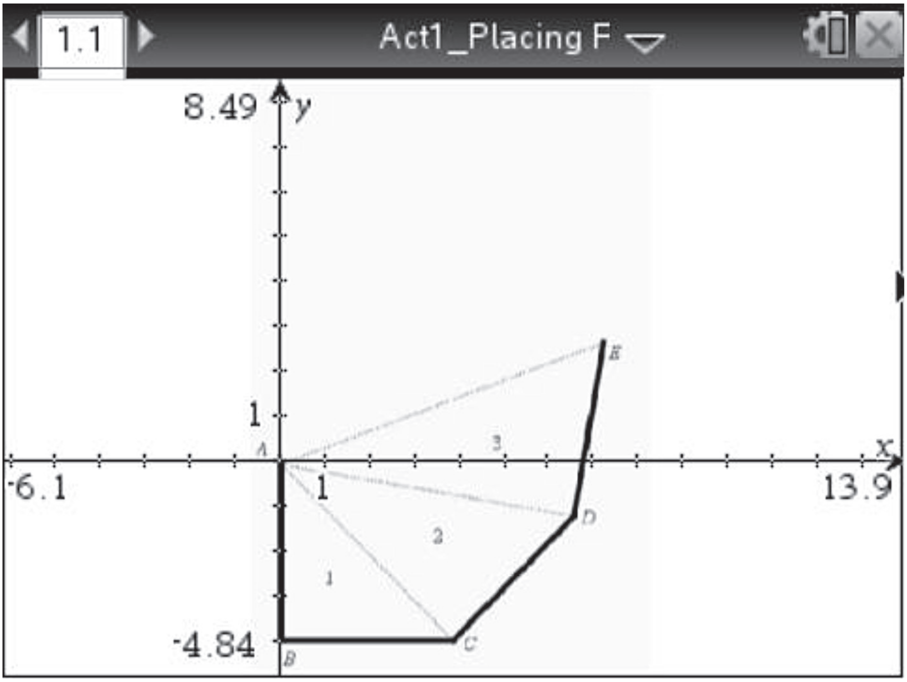

Figure 2. Placing a worksheet image in the Nspire Graphs application

The class activity, replacing pencil and ruler with handheld technology, has now become

- Placing point F with a visual estimate on the handheld screen,
- Measuring $\overline{AF}$ and $\angle AEF$ to refine that estimate, and
- Observing and critiqueing any selected student's work-in-progress being displayed live for all to see.

Every student with a handheld device is now a potential public contributor to the learning process, and knows the teacher is but a click away from handing over the main display to that student. The teacher's professional judgement can thus improve students' "affective engagement" (Pierce et al, 2007) and minimise the disengagement experienced by many.

Using a QuickPoll to discuss Algebraic Equivalence

Activities 2 and 3 of the lesson involve students exploring the ever-increasing perimeters of the first 13 triangles in the Surd Spiral, expressed in exact surd format. These are generated in a Lists & Spreadsheets application after a formula for the perimeter of the *n*th triangle is discovered to be $(1+\sqrt{n}+\sqrt{n+1})$ units. Questions 9 through 12 in Activity 4 of the lesson then require students to investigate the (assumed positive) *difference* in the perimeters of various pairs of specific successive triangles, in establishing whether or not that difference is constant.

Question 13 of the original version of the lesson then asks students to then generalise that difference for any two adjacent traingles in the Spiral. Again, there is an expectation of pencil and paper working out leading to the simplified result of $(\sqrt{n+2}-\sqrt{n})$ units, although CAS features of the Calculator application could be used at the teacher's discretion.

An opportunity now exists for students to do that work as before, but then to electronically submit their result to the teacher through the Question application (here, an Expression type) formatted as a QuickPoll (see Figure 3, below).

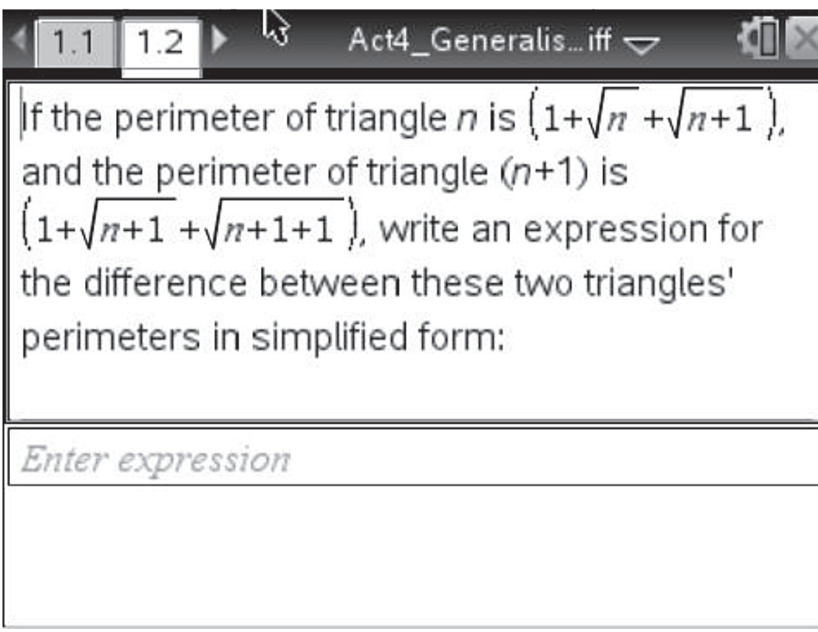

Figure 3. QuickPoll for student responses to Activity 4, Question 13

The pedagogical opportunity here for teachers is quite interesting. After all expressions are submitted, they can be displayed as data in a bar graph, which will show the various responses and how many students submitted each variation – which could be quite a wide range in some cases. The correct (as formatted ahead of time by the teacher) expression of $(\sqrt{n+2}-\sqrt{n})$ will stand out in a different colour, and the teacher has the option of also activating any equivalent expressions such as $(1+\sqrt{n+1}+\sqrt{n+2}-1-\sqrt{n}-\sqrt{n+1})$. These can be compared to the near-misses such as $(1+\sqrt{n+1}+\sqrt{n+2}-1+\sqrt{n}+\sqrt{n+1})$; the processes of simplification and correct use of brackets can then be discussed. Any student's working out done on the Calculator screen can also be viewed in the manner described earlier. Thus, the range of student responses can be analysed by those same students.

Using Multiple-Choice to Discover Patterns and Lead to Proof

In Activity 5 of the lesson students turn their attention to investigating area rather than perimeter, discovering that the area of triangle n is $\frac{\sqrt{n}}{2}$ units^2. Question 24 of the original lesson asks them to find which triangle has an area 100 times that of Triangle 100; the correct answer is Triangle 1 000 000.

There are, in fact, an infinite number of pairs of Surd Spiral triangles which have this 100:1 ratio of their areas. The algebraic proof of this fact is neat, concise and achievable either with or without CAS assistance, but not for many Year 10 students. Nspire's Question application using the Multiple-Choice feature (where more than one option can be correct) can be employed as a QuickPoll as shown in Figure 4 below (note the computer view of the Question application to be able to see the complete text):

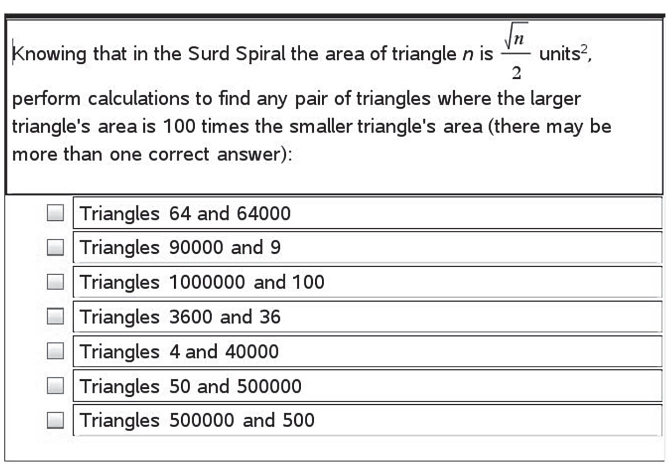

Figure 4. Multiple-Choice QuickPoll to promote investigation of patterns

The format of this Multiple Choice item implies a trial-and-error approach will be sufficient to answer the question successfully, providing a somewhat safer environment for hesitant students than an open response question. Students who perform the calculations correctly will discover that four of the seven pairs satisfy the required area ratio. There will be some who choose incorrect pairs and others who do not find all four correct pairs. As in the previous example, when the teacher displays the results of the QuickPoll the correct answers can appear highlighted. Discussion can then be directed towards finding the linking feature between triangle numbers (a ratio of 10000:1), the infinite nature of the solutions, and the associated algebraic proof for more able students.

Conclusion

Providing an environment which attracts and maintains students' attention and engagement is a constant challenge facing secondary mathematics teachers. Technology such as TI-Nspire Navigator invites a deeper level of participation from students than might have been possible in years past, and there are plenty of new activities written for these emerging technologies.

The examples shown in this paper have been presented to encourage experienced teachers in particular to look also at those existing lessons, worksheets and activities from their earlier years of teaching. If they can be reformatted to include opportunities for increased student engagement and for better teacher monitoring of student progress during lesson time rather than at the conclusion of the unit of work, improvement in attitudes towards and outcomes in mathematics learning might be possible for more of our students.

References

HREF1 The University of Melbourne (2009). Retrieved from http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/general_access/curriculum_resources/surd_delights/index.shtml#SurdNspire

HREF2 The Mathematical Association of Victoria (2009). Conference content. Retrieved from http://www.mav.vic.edu.au/files/conferences/2009/MAV-Synopsis-09.pdf

Pierce, R., Stacey, K., & Barkatsas, A. (2007). A scale for monitoring students' attitudes to learning mathematics with technology. *Computers and Education, 48,* 285-300.

Stacey, K., & Price, E. (2005). Surds, spirals, dynamic geometry and CAS. In J. Mousley, L. Bragg & C. Campbell (Eds.), *Mathematics – celebrating achievement. Proceedings of the Mathematical Association of Victoria 43rd Annual Conference* (pp. 298-306). Melbourne, Australia: Mathematical Association of Victoria.

Turner, J. C., & Patrick, H. (2004). Motivational influences on student participation in classroom learning activities. *Teachers College Record, 106*(9), 1759-1785.

EXPLORING GAMES IN MATHEMATICS

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Children are naturally interested in games as they depict a sense of fun while they are immersed in them. The experience of incorporating games in a school will be shared and how some of these selected games can be linked to mathematics will be discussed. A literature review on the impact of playing games and mathematics will also be shared.

Introduction

Board games have been around since BC 3500 and are still around. The Royal Game of Ur, the oldest complete set of gaming equipment ever found, the Senet appears in Egyptian dynastic history through the 4th century B.C. to the more familiar games like Snakes and Ladders in India in the 1200's and Checkers in the 1500's (History of Games Timeline, 2001). However the popularity of board gaming began in the early 1960s and Hasbro, the world's largest gaming manufacturer, reported record sales of board games in 2003. In 2004, it was deemed as a Golden Age of gaming (Shapiro, 2004).

Hence a neighbourhood primary school with pupils from grades 1 to 6 in Singapore began to explore the implementation of games to engage the pupils both in and out of the classroom. Pupils in these grades are from 7 to 12 years old and hence interested in games. The games aim to develop their interest, beliefs, appreciation, confidence and perseverance under the attitude component in the mathematics framework. They also develop numerical calculation, spatial visualization skills, monitoring and self-regulation of learning and reasoning, communication and connection in the process component in the mathematics syllabus (Curriculum Planning and Development Division, 2006).

The theme 'Mathematics Everywhere' was chosen to represent the pervasiveness of mathematics learning through games in the school. It is also hoped that through games, pupils will enjoy their learning in mathematics and more importantly develop crucial values like losing graciously, cooperation, taking turns and responsibility for maintaining the games.

Review of Literature

Ernest (cited in Ainley, 1990) claimed that games could teach mathematics effectively in four ways namely providing reinforcement and practice of skills, providing motivation, helping the acquisition and development of concepts, and developing problem-solving strategies. Ernest argued that motivation and attitudes are the likely reasons for the success of games in mathematics. However, games are often used as a reward for those who have completed their work earlier and thus the pupils who need the games to motivate them in their learning would get the least (Ainley, 1990). Thus games should be incorporated into the school curriculum and allow all the pupils to have access to them. Games can assist pupils in learning the processes in mathematics, predicting and testing, conjecturing, generalizing and checking and justifying (Ainley, 1990). These processes can be achieved by carefully selecting games which will lead pupils to demonstrate and practice these processes. In the game Crossing all these processes could be incorporated in the game (Ainley, 1990). The game only requires a board which resembles a chess board and a 1-6 die. With that, 2 – 3 players can be engaged meaningfully in having fun and yet learn the processes in mathematics.

Harkness and Lane (2012) also find that the processes of specializing, conjecturing, generalizing and convincing could be incorporated into games. They illustrated these in three game shows – *Survivor*, *The Biggest Loser*, and *Deal orNo Deal*? However they find that data indicated that students did not engage in the process of mathematical thinking unless directed to do so. Thus this illustrates the importance of deliberately planning for the thinking process to occur when pupils are playing games.

Civil (2002) explores the three different kinds of mathematics namely school mathematics, mathematicians' mathematics in the school context and everyday mathematics and uses games to study the impact of games on these three kinds of mathematics. The *Game of Nim* is an example (Civil, 2002). This game involves a single pile of 12 pieces and two players. Each player takes 1, 2 or 3 pieces, and the player who takes the last piece is the winner. Hence pupils may play the game at random with no specific strategy in mind. But the teacher can plan for the pupils to find a winning strategy in this game, to leave the opponent with the last 4 pieces. It is found that pupils participated in the game when it was related to everyday mathematics but withdrew when it progressed to more formal mathematics. It is also found that pupils who seemed to be uninterested in academic mathematics developed positive attitude in the everyday mathematics started discussing and even helped another pupil. In the process, questioning and conjecturing took place and technical vocabulary and definitions were investigated through the discussions. Thus games could be an effective tool to engage pupils in the learning of mathematics but if the emphasis is on the school mathematics, then pupils may withdraw quickly as they may deem the games to be another academic learning exercise which the pupils are already avoiding. Hence games should be seen as an extension instead of a direct replacement of what is taught in a formal mathematics lesson for it to engage the pupils.

So what are games? Bright, Harvey and Wheeler (1985) listed 7 criteria:

1. A game is freely engaged in.
2. A game is a challenge against a task or an opponent.
3. A game is governed by a definite set of rules.
4. Psychologically, a game is an arbitrary situation clearly delimited in time and space from real-life activity.
5. Socially, the events of the game situation are considered in and of themselves to be of minimal importance.
6. A game has a finite state-space (Nilsson, 1971). The exact states reached during play of the game are not known prior to beginning of play.
7. A game ends after a finite number of moves within the state-space.

Bright, Harvey and Wheeler's comprehensive list of criteria addresses the various domains of learning from the psychological perspective to the social domain. Games will help pupils to exercise the spatial visualization skill in the skills component in the syllabus (CRPP, 2006). As the pupils play the games, they will have to visually consider the moves and the implications behind each move before deciding to act. Thus games are in themselves attractive to pupils and hence pupils will want to be engaged in the games. It is up to the teachers to devise and structure the games to develop the various aspects in the learning of mathematics; attitude, processes, mathematics contents, skills and metacognition as indicated in our mathematics framework. (CRPP, 2006)

Mathematics Everywhere

The first consideration is to decide on the types of games that will meet the needs of the pupils. The second consideration is the progression of these games so that pupils will experience success and development in the process, skills, metacognition, concepts and attitude. The final consideration is the time and space for the teachers and pupils in implementing the games.

The fundamentals in learning mathematics are looked into and games that meet these basic requirements are implemented. Pupils need to build up the multiplication tables of 2, 3, 4, 5, 6, 7, 8 and 9 and commit to memory by grade 3. Hence a traditional game of hop-scotch is designed and painted on the floor in the canteen for the pupils to play. In the process of hopping from one box to another, they have to recite the multiplication tables slowly forward and backward as they hop forward and backward to complete the game. Pupils could be involved in the game of hop-scotch during recess or after school while they are waiting for the school bus or another school activity. Teachers could also arrange for some of the pupils to go to the canteen to practice the multiplication tables while the teacher conducts a separate activity for another group of pupils.

Another foundation is the four operations where we would like the pupils to calculate the addition, subtraction, multiplication and division of 2-digit numbers mentally. Hence a 'Open & Close' game is sourced and mounted on a wall along a walkway. In this game, pupils have to open a card which will reveal an answer or a question. The pupil will have to open another card to match the answer or the question. If they match, the pupil scores one point. If the cards do not match, the pupil will turn over the cards and the next pupil continues with the game. Thus the pupils have to know their addition, subtraction, multiplication and division operations well. They will also need to apply the skills learnt like count on, near doubles, make tens, partial products to calculate efficiently. Memory will always be needed in any learning. Hence by turning over the cards, the pupils will have to remember the positions of the various cards which have to be turned over because their peers do not get the matching cards. That will help them to understand the position of each card as well as the relative positions of two or more cards. The vocabulary used will be dependent on the levels of the pupils. The pupils may use

top, bottom, left , right initially and progress to north, south , east , west later. Thus there are multiple opportunities for pupils to develop the learning in mathematics. An extension of the activity will be to engage pupils to look at the relationship between the operations and within the operations. What do you notice when I add 2

numbers and multiply the same 2 numbers? Explain. Is it always true that when you multiply two numbers as compared to adding two numbers, the answer will be greater? 2 x 3 = 6 is greater than 2 + 3 = 5. 2 x 1 = 2 is not greater than 2 + 1 = 3. Hence pupils do make conjecture and generalize before they verify with a counter example to show that it is not true all the time. What do you notice when I add 2 numbers and reverse the order of the numbers? Explain. 2 + 3 = 5 and 3 + 2 = 5 will lead the pupils to understand that addition is commutative. The process of making a conjecture, generalizing and then verifying with sufficient examples but not a formal proof at the primary level will help pupils in the process in mathematics. Is that true for subtraction? Why? By structuring the learning either in the form of questions posed by the teachers or having 'Game leaders' who will man the games and able to extend the learning of the pupils, the pupils will be able to progress and reinforce their learning at a pace that is comfortable to them.

These games are carried out outside the classrooms. Another set of games are carried out in the classrooms. They are structured by levels and each level will have a set of games to engage them. Some examples are listed in Figures 1 and 2 below.

Math Games Inventory List (Primary 2)

Class: 2 / ____

Name of Game	Quantity
Smart Driver	
Tortoise Chess Game	
Doctor's Ball Game	
Chameleon Game	
Line up 4	
Connect-a-cube (100 pieces)	
Total number of items	

Figure 1. Games for a grade 2 class

Math Games Inventory List (Primary 3)

Class: 3 / ____

Name of Game	Quantity
Perilous Single Plank Bridge Game	
Tortoise Chess Game	
Frog Chess Game	
Line Up 4	
Puzzle Cards	
Stereography	
Tower of Hanoi	
Red and Black Cubes	
Tangram	
Doctor's Ball Game	
Total number of items	

Figure 2. Games for a grade 3 class

These games are played before school once a week and a double period in mathematics is planned in the timetable each week for games. Thus teachers and pupils have a structured and planned time to play the games. Pupils are required to understand the rules of the games which are similar to the rules that pupils need to adhere to when they do mathematics. Then the pupils are supposed to play the games to win and then reflect on the strategies that help them to win. Hence a play stage will lead them to be more conscious of the pattern in a winning strategy. That will indirectly lead them to the process of conjecturing, looking for a pattern to generalize and then playing the game using the strategy to verify if their conjecture is correct.

Some Findings

Do student teachers and pupils believe that games do help pupils in their learning? Are games the choice of the student teachers and pupils in teaching and learning? The findings indicate that for student teachers and pupils to incorporate games in their lessons, much work remains to be done to convince both parties that games can be another resource to tap on. The 25 student teachers are in their first year post graduate diploma study and 39 pupils in grade 6 in a neighbourhood school.

The findings are shown in Table 1 below.

Table 1. Ranking of Preferences of Pedagogies

Pedagogies	Student Teachers		Pupils	
	Personal	Class	Personal	Class
Game	3rd	4th	3rd	4th
ICT	2nd	3rd	1st	2nd
Story	5th	5th	5th	3rd
Cooperative learning	4th	2nd	2nd	1st
Manipulative	1st	1st	4th	5th

Five pedagogies are selected namely game, information and communication technology (ICT), story, cooperative learning and manipulative. In Table 1, the personal preference of using game is the student teachers' third choice and if they were to use game in the class, it drops to the fourth choice. There is a close match of their personal preference to what they would actually do in class. Comparing the expectations of the pupils, personally the pupils would rank the use of game as their third choice which is aligned to the student teachers' preference. In class, the pupils rank game as the fourth choice and it is aligned to that of the student teachers. Overall, game is not the student teachers nor the pupils top choice. This could be due to the difficulty of relating game directly to the learning in mathematics. Also game is not commonly associated in mathematics. Manipulatives and the use of ICT are more commonly advocated in schools due to the mass roll out of manipulatives to all schools for grades 1 and 2 pupils and the masterplan 3 for ICT in schools in 2009. Hence manipulative figured strongly as the first choice for the student teachers for both personal and in class. This could be strengthed by the emphasis during their training that manipulatives would be helpful to help the pupils in the concrete stage of learning. However for the pupils, ICT and cooperative learning are their top two choices personally and in class. This could be that they are already in grade 6 and hence are able to think in the abstract stage.

Conclusion

The use of games has opened up another possibility in the teaching and learning in mathematics. There is still much work to be done to convince teachers and pupils of the values that games can contribute to the learning in mathematics.

Games would need to be structured and planned carefully with both a playfulness and exploratory aspect in learning before progressing to the process and mathematical contents. This is to ensure that pupils do not view games as another exercise to drill the pupils in the learning of mathematics. The openness of using game is also an asset in the skillful hands of the teachers or student game leader. By being observant in what the pupils are doing when they are playing, the teachers and student game leader can lead and extend the learning of the pupils in the midst of playing the games. Since games are all around us, so why not tap on them and let the pupils have fun and learn mathematics at the same time.

References

Ainley, J. (1990). *Playing games and learning mathematics.* In L. P. Steffe & T. Wood (Eds.), Transforming children's mathematics education: International perspectives (pp. 84-91). Hillsdale, NJ: Erlbaum.

Bright, G. W., Harvey, J. G., & Wheeler, M. M. (1985). *Learning and mathematics games.* Journal for Research in Mathematics Education. Monograph, Vol. 1

Civil, M. (2002). *Everyday mathematics, mathematicians' mathematics, and school mathematics: Can we bring them together?* Everyday and academic mathematics in the classroom, (pp. 40-62). Reston, VA: National Council of Teachers of Mathematics.

Curriculum Planning and Development Division. (CPDD), (2006). Mathematics syllabus primary. Singapore: Ministry of Education.

Harkness, S. S., & Lane, C. P. (2012). *Game show mathematics: Specializing, conjecturing, generalizing, and convincing.* The Journal of Mathematical Behaviour, 31, 163-173.

History of games timeline. (2001). Retrieved from http://historicgames.com/gamestimeline.html

Shapiro, D. (2004). *A brief history of gaming.* The Games Journal. Retrieved from http://www.thegamesjournal.com/articles/BriefHistory.shtml

THE BEST WAY TO TEACH MATHEMATICS

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There is no one best way to teach mathematics. This depends primarily on the learner, as it is their maths. The teacher needs to motivate, interest, know and value the learner. The learner's talents need to be identified, nurtured and developed. Teachers generally have to provide this personalised learning within groups of students in a classroom. To maintain this working relationship, a variety of learning activities to develop understanding, rather than knowledge, are desirable to meet the various needs within a group of students.

Gain and Maintain Learner Interest

"You can lead a horse to water, but you can't make it drink." You can try to teach, but it is the learners who determine if you succeed. It is not possible to teach uninterested or distracted students. An essential first step is to gain student interest and a harder second step is to maintain it. Depending upon the student cohort there are many ways to gain student interest.

The first impression of you from a class is often the most lasting. Those fortunate to have a name consisting only of the letters 8, E, 9, h, I, L, O and S are able to introduce themselves by getting students to multiply the prime factors of their name turned upside down on a calculator e.g. multiply three by seven by thirteen cubed gives 46137 which when turned upside down spells LEIGh. Introductory comments that can gain your students' attention may be along the lines of, "If you don't follow what I'm teaching then tell me to lift my game. This way you'll make me a better teacher" and "The best teachers are often sitting beside you. When your classmate says it is easy and you can't follow, he is not implying you are dumb, but is indicating that once you get the hang of it, all is plain sailing".

Stories about past students can provide encouragement. Two past students at a secondary school regularly got into trouble playing computer games when they were not supposed to. After students were asked, what they thought became of these past students, they were surprised to learn that these past students now part-own the largest computer games company in the world, Valve Software, and probably earn more in one year than a teacher does in their entire career. Students were informed that they were top maths students. Another past student, from the same small town as the other two, was Australia's sole representative in the Men's Mountain bike event at the Beijing and London Olympics. He was not a top maths student, indicating that it is not necessary to excel in maths to be successful in life.

Younger students can be fascinated by 'magic' tricks. A much used pack of Stuthards 'Stripmark' Marked Cards (specially prepared marked cards for magic tricks) has proved invaluable. Students are impressed to see a deck of seemingly all red cards magically change to all black cards and later alternating red and black. When a selected card is replaced and the deck shuffled, they are amazed to see the teacher draw out their selected card without looking at it, then tell them what card it is. Explaining how these tricks are done enables the teacher to move onto other areas.

For older students Youtube has material to engage the entire class especially if a multimedia projector, reasonable speakers and a darkened classroom are available. Some with some mathematics relevance are: *Mr Bean – Counting Sheep; Ma & Pa Kettle Math; Abbott And Costello 13×7 is 28; 1959 – Donald Duck – Donald in Mathmagic Land.* Those with little or no mathematical relevance but some cycling relevance are: *Inspired Bicycles – Danny MacAskill April 2009; Commercial – Greene King IPA Beer; Big Slip n Slide into water; Extreme Mountain Bike Crash with 170 kph; Bicycle Competition.*

Cooperation and competition motivate some students. A multiplication sheet similar to that shown in Figure 1 is handed to each student. Students write their name on their sheet. The top row continues across to 90 × 90, the first column continues down to 1 × 19 and the ninth column tenth row has 81 × 99.

$$\begin{array}{rrrrl} 10\times10= & 20\times20= & 30\times30= & 40\times40= & \ldots \\ 9\times11= & 19\times21= & 29\times31= & 39\times41= & \ldots \\ 8\times12= & 18\times22= & 28\times32= & 38\times42= & \ldots \\ 7\times13= & 17\times23= & 27\times33= & 37\times43= & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Figure 1. Multiplication Race

The last three rows are similar to Figure 2. Students are told there is a pattern in the answers; they are allowed to use calculators and can work together but each student must complete their own sheet. The teacher will provide an answer if asked. Products of mixed numbers must be written as mixed numbers and products of decimals must be written as decimals.

$$\begin{array}{rrrrl} 1\tfrac{1}{2}\times1\tfrac{1}{2}= & 2\tfrac{1}{2}\times2\tfrac{1}{2}= & 3\tfrac{1}{2}\times3\tfrac{1}{2}= & 4\tfrac{1}{2}\times4\tfrac{1}{2}= & \ldots \\ 1\cdot5\times1\cdot5= & 2\cdot5\times2\cdot5= & 3\cdot5\times3\cdot5= & 4\cdot5\times4\cdot5= & \ldots \\ 15\times15= & 25\times25= & 35\times35= & 45\times45= & \ldots \end{array}$$

Figure 2. Last three rows of Multiplication Race

Experience has shown that virtually all students participate eagerly in this seemingly uninteresting task. Few, if any, discover or are aware of the numerical application of the difference of two squares. e.g.

$$\begin{aligned} 37\times43 &= (40-3)(40+3) \\ &= 40^2-3^2 \\ &= 1600-9 \\ &= 1591 \end{aligned}$$

$$\begin{aligned} \left(4\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2 &= \left(4\tfrac{1}{2}-\tfrac{1}{2}\right)\left(4\tfrac{1}{2}+\tfrac{1}{2}\right) \\ \left(4\tfrac{1}{2}\right)^2-\tfrac{1}{4} &= 4\times5 \\ \left(4\tfrac{1}{2}\right)^2 &= 20\tfrac{1}{4} \end{aligned}$$

Figure 3. Arithmetic application of the difference of two squares

Most learn that $\frac{1}{2}=0\cdot5$ and $\frac{1}{4}=0\cdot25$ if they did not know this beforehand, and, can generalize from $\left(4\frac{1}{2}\right)^2=20\frac{1}{4}$ to $4\cdot5^2=20\cdot25$ and $45^2=2025$ as well as gaining some tables practice. This activity also provides an opportunity to explain the operation of the fraction key, $a\frac{b}{c}$, found on many school calculators.

Get to Know Your Students

Successful teaching requires that each student believes you care about them and their learning. It is important to identify and note specific skills, talents, qualities in each student e.g. logical, reliable, neat, fast, accurate, cooperative, competitive, considerate, creative, original, persistent, intelligent, honest, helpful, etc. A teacher needs:

> *the ability to really see their students, their potential for brilliance, the essence of their humanness through the somewhat obscuring clouds of sometimes distracting behaviour. A practised ability to see the good that may be dormant in a student, waiting for someone to give permission for that goodness to explode into existence, is the essence of a great teacher. (Beadle, 2011, p. 29).*

Carefully correcting an individual student's work with them beside you enables a teacher to get to know a student. However, this can be very time consuming and requires the rest of the class to be fairly quiet and meaningfully occupied. Although the teacher learns a lot about a student through correcting their work, the student learns little or nothing if not present. This is generally true, as most correction is done away from the class and often outside school. A lot of time can be saved and students can learn by having the whole class correct their work. Of course, some students may add or change answers but this generally does not matter if results are not recorded or critical to student assessment. Valuable learning opportunities often occur during tests. The teacher is sometimes faced with the dilemma; should they let a learning opportunity pass or should they assist and invalidate the test results. Assisting during a NAPLAN test is not advisable for those wishing to continue their teaching career but for class tests, assisting student learning may be the best course of action.

A worthwhile activity can be to go through these tests, especially the multiple choice questions suggesting alternative correct answers e.g. $1+1=$ a) $\boxplus$ window b) 11 house number c) 10 base 2. If these suggestions appear far-fetched: consider a two part test result given as $\frac{50}{60}+\frac{30}{40}=80\%$ when mathematically correct results are $\frac{50}{60}+\frac{30}{40}=158\frac{7}{12}\%$ or $\frac{50}{100}+\frac{30}{100}=80\%$. The first result conveys more information, the second is meaningless in regards to the test and students find the last frustrating if they are unaware of the marks available in each section.

Engage Students

Euclid is said to have replied to King Ptolemy's request for an easier way of learning mathematics that "there is no Royal Road to geometry" (HREF1). Maths teachers face an apparent daunting task in getting students to struggle hard to learn.

The conference theme, 'It's My Maths: Personalised Mathematics Learning', hits the nail on the head when it comes to the best way to teach mathematics, especially since what works for one student does not necessarily work for another. However, providing every student with an individualised learning program is generally not possible. Individual learning styles may be best addressed by a variety teaching styles and content.

Providing tasks on which all students can profitably participate is key to engaging every student. Problem solving tasks often meet this requirement. The following problem is suitable for a wide range of student abilities and ages from middle primary to senior secondary.

A farmer finds that emus have become mixed in with his sheep in a paddock. He counts 30 heads and 100 legs. How many sheep are there in the paddock? *Hint:* For every emu head there are 2 legs whilst for every sheep head there are 4 legs.

Equations

let e be the number of emus

let s be the number of sheep

$$\begin{aligned} e + s &= 30 \quad \ldots(1) \\ 2e + 4s &= 100 \quad \ldots(2) \end{aligned}$$

Figure 4. Simultaneous equations

Elimination Methods

$$\begin{array}{ll} 2\times(1) & 2e + 2s = 60 \ldots(1)' \\ & 2e + 4s = 100 \ldots(2) \\ (2)-(1)' & 2s = 40 \\ & s = 20 \\ & \text{There are 20 sheep} \end{array}$$

$$\begin{array}{ll} 4\times(1) & 4e + 4s = 120 \ldots(1)' \\ & 2e + 4s = 100 \ldots(2) \\ (1)'-(2) & 2e = 20 \\ & e = 10 \\ & \text{Sub. 10 for } e \text{ in } (1) \\ & 10 + s = 30 \\ & s = 20 \\ & \text{There are 20 sheep} \end{array}$$

$$\begin{array}{ll} (2)\div 2 & e + 2s = 50 \ldots(2)' \\ & e + s = 30 \ldots(1) \\ (2)'-(1) & s = 20 \\ & \text{There are 20 sheep} \end{array}$$

Substitution Methods

$$\begin{aligned} s &= 30 - e \ldots(1)' \\ &\text{Sub. } 30 - e \text{ for } s \text{ in} \ldots(2) \\ 2e + 4(30 - e) &= 100 \\ 2e + 120 - 4e &= 100 \\ -2e &= -20 \\ e &= 10 \\ &\text{Sub. 10 for } e \text{ in } (1) \\ 10 + s &= 30 \\ s &= 20 \\ &\text{There are 20 sheep} \end{aligned}$$

$$\begin{aligned} e &= 30 - s \ldots(1)' \\ &\text{Sub. } 30 - s \text{ for } e \text{ in} \ldots(2) \\ 2(30 - s) + 4s &= 100 \\ 60 - 2s + 4s &= 100 \\ 2s &= 40 \\ s &= 20 \\ &\text{There are 20 sheep} \end{aligned}$$

Figure 5. Simultaneous equations can be solved in a variety of ways

Graph Method

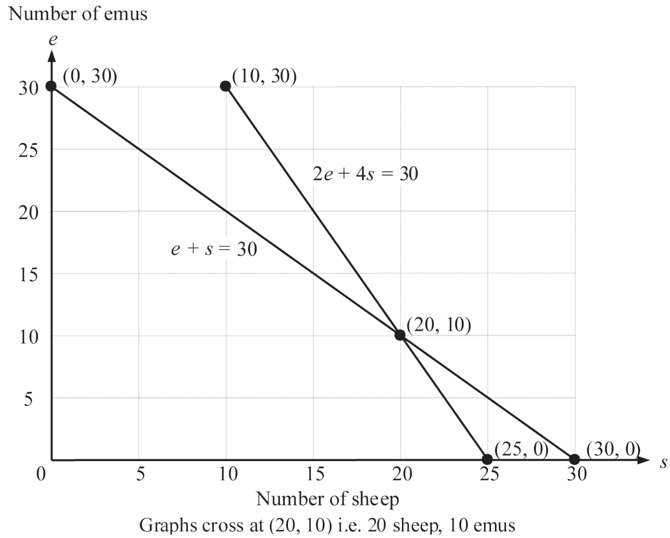


Figure 6. Graph solution (requires some point plotting)

Matrix Method (detailed)

$$\begin{bmatrix}1 & 1\\2 & 4\end{bmatrix}\begin{bmatrix}e\\s\end{bmatrix}=\begin{bmatrix}30\\100\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\2 & 4\end{bmatrix}^{-1}\begin{bmatrix}1 & 1\\2 & 4\end{bmatrix}\begin{bmatrix}e\\s\end{bmatrix}=\begin{bmatrix}1 & 1\\2 & 4\end{bmatrix}^{-1}\begin{bmatrix}30\\100\end{bmatrix}$$

$$\tfrac{1}{2}\begin{bmatrix}4 & -1\\-2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\2 & 4\end{bmatrix}\begin{bmatrix}e\\s\end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}4 & -1\\-2 & 1\end{bmatrix}\begin{bmatrix}30\\100\end{bmatrix}$$

$$\tfrac{1}{2}\begin{bmatrix}4\times1+-1\times2 & 4\times1+-1\times4\\-2\times1+1\times2 & -2\times1+1\times4\end{bmatrix}\begin{bmatrix}e\\s\end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}4\times30+-1\times100\\-2\times30+1\times100\end{bmatrix}$$

$$\tfrac{1}{2}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}e\\s\end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}120-100\\-60+100\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}e\\s\end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}20\\40\end{bmatrix}$$

$$\begin{bmatrix}1\times e+0\times s\\0\times e+1\times s\end{bmatrix}=\begin{bmatrix}10\\20\end{bmatrix}$$

$$\begin{bmatrix}e\\s\end{bmatrix}=\begin{bmatrix}10\\20\end{bmatrix}$$

Figure 7. A matrix solution (grey working not necessary once method understood)

TI-nspire CAS

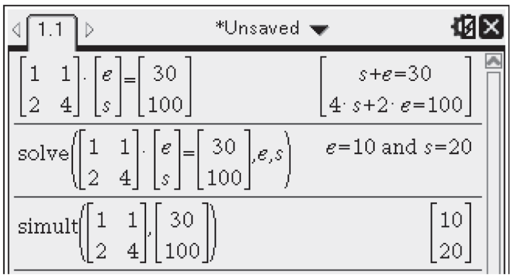


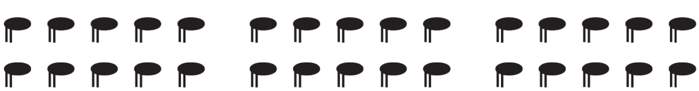
Figure 8. CAS calculator solutions

These methods may be beyond many students however the following methods solve the problem.

Primary Schoolgirl Method

If all the sheep stood up on their back legs there would be 60 legs on the ground (2 legs for each head). The extra 40 legs must be sheep front legs in the air, so there are 20 sheep (1 sheep for each pair of legs in the air).

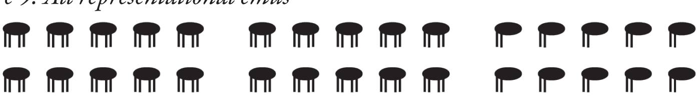
Picture Method



Draw extra pairs of legs changing emus to sheep until 100 legs

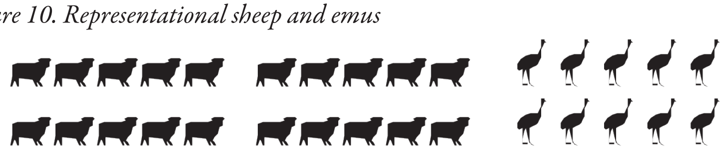
30 emus, 0 sheep, 60 legs

Figure 9. All representational emus



10 emus, 20 sheep, 100 legs

Figure 10. Representational sheep and emus



10 emus, 20 sheep, 100 legs

Figure 11. Realistic sheep and emus

Whilst younger students might enjoy this problem and its ridiculousness, some older students may not appreciate the humour and see it as pointless and unrealistic. If the problem is changed to: *'Tickets to a school function are \$2 for students and \$4 for adults. A total of 30 tickets are sold for \$100. How many adults have tickets to the function?',* then these students may see the problem as more meaningful. Creative students can be challenged to come up with different scenarios using the same and different numbers. Bicycles and tricycles is one such variation.

Varied Activities

The following activities are a sample designed to teach important concepts whilst engaging most if not all students.

Drawing

Most students enjoy drawing and this is a necessary skill in mathematics. A simple challenge to students is to draw a horizontal 10cm line, a 6cm line from the left end and an 8cm line from the right end such that the lines form a triangle with no shortfall or overlap i.e. the sides of the triangle are 10cm, 6 cm and 8 cm. For older trade oriented students these measures can be given as 100mm, 60mm and 80mm. Many students cannot believe that with a glance, the teacher (who knows this forms a right angled triangle) can tell if a student has met this challenge successfully. The teacher often needs to prove this through use of a ruler. When one of the few students to use a drawing compass to perform this task, was asked how he knew to do this, he explained that his metalwork teacher had shown him. Many students learn more mathematics in practical subjects than in mathematics, making mathematics teachers with a practical subject method, invaluable.

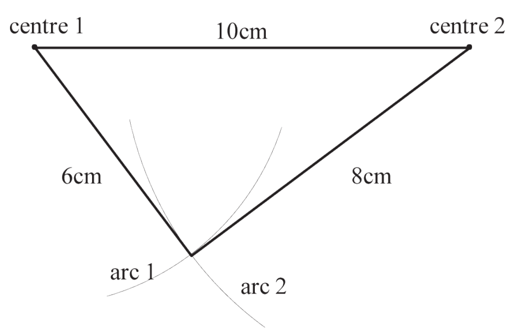


Figure 12. Triangle constructed using drawing compass

By mistake, students were once asked to draw a triangle with sides of 10cm, 6cm and 3cm. After realising that it could not be drawn students came up with their version of *'any two sides of a triangle together are greater than the third'.*

3, 4, 5 Triangle

With the help of two others hold a measuring tape at 3 metres, 7 metres and 12 metres (Note 3m + 4m = 7m and 7m + 5m = 12m. Hold the 12 metre mark and the start of the tape together and pull the tape taut. The angle between the two shorter sides (i.e. at 3m) is a right angle. Make a larger right angled triangle by holding the tape at 6 metres, 14 metres and 24 metres. Knowing that a triangle, with sides in the ratio of 3 : 4 : 5, is a right angled triangle is a very useful practical skill. Further, knowing that a parallelogram with equal diagonals is a rectangle enables accurate right angles to be constructed in practice.

Proof of Pythagoras Theorem

To understand this proof students need to be familiar with area (the amount of surface) and realise that the area visible does not change as a shape is moved around on it.

Cut four identical right angled triangles and arrange them in a square as shown in Diagram 1 below. The part of the square uncovered by the triangles is a square with side length the same as the hypotenuse length. This area is equal to the hypotenuse length squared i.e. c^2. Keep the triangles within the square and move triangle 3 up and right as shown in Diagram 2, then move triangle 4 left as shown in Diagram 3, then move triangle 1 down as shown in Diagram 4. Now the part of the square uncovered by the triangles is two squares with side lengths the same as the two shorter sides. This area is equal to the sum of the squares of these sides i.e. $a^2 + b^2$. Since the amount of square uncovered is the same $a^2 + b^2 = c^2$.

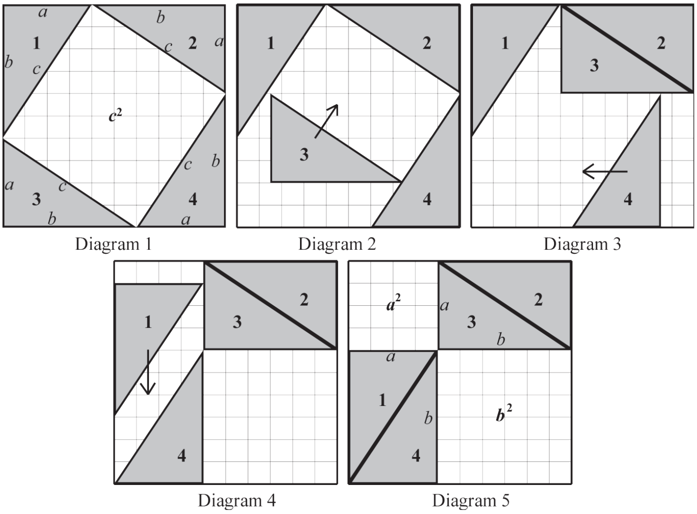


Figure 13. Proof of Pythagoras Theorem

The drawing compass

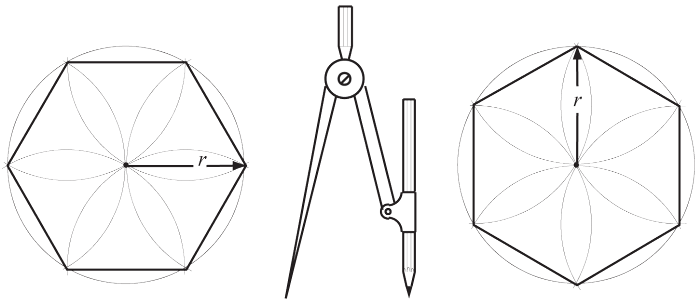


Figure 14. Hexagonal 'flowers' constructed with compass and ruler

An often underutilised drawing instrument is the compass. Besides being useful in constructing perpendicular bisectors, angle bisectors and angles of 60°, 120°, 90°, 30° and 45° it is relatively easy to construct hexagons (and hexagon flowers). Students readily see that the distance around the hexagons in Figure 14 is six times the radius. As the distance around the circle is a little longer, students can appreciate that $C > 6r, \quad C > 3 \times 2r$. Now as $d = 2r, \quad C > 3d$. As $\pi > 3$, the formulas $C = \pi d$ and $C = 2\pi r$ become more meaningful.

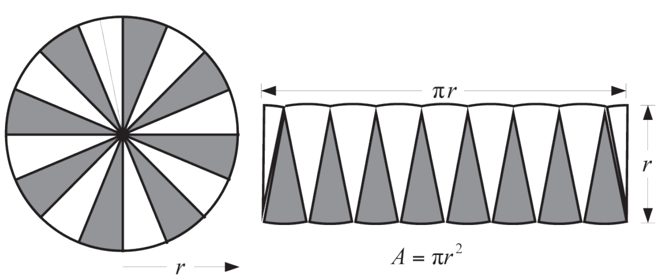


Figure 15. Area of circle and 'rectangle' are equal

The area formula $A = \pi r^2$ makes sense when students cut the sectors out and rearrange them into a 'rectangle'. For those who wish to compute π the series $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right)$ usually suffices, even though it converges ever so slowly. Using a spreadsheet and averaging the oscillating values quickly gives an accurate value for π.

Table 1. Spreadsheet formula with inputs to obtain an accurate π approximation.

	A	B	C	D
1	1	1	4	
2	=-A1	=B1+2	=C1+4*A2/B2	=(C1+C2)/2

Only seven cells need to be filled in to use this series to evaluate an approximation for π. Cell D2 can be copied to E3, F4, G5, H6, I7 and so on diagonally down to the left. Row 2 can be highlighted from cell A2 to cell D2 then filled down. Cells E3, F4, G5, H6, I7 and so on can be filled down.

Table 2. Spreadsheet normal view showing converging values for π.

	A	B	C	D	E	F	G	H	I
1	1	1	4						
2	-1	3	2.66667	3.33333					
3	1	5	3.46667	3.06667	3.20000				
4	-1	7	2.89524	3.18095	3.12381	3.16190			
5	1	9	3.33968	3.11746	3.14921	3.13651	3.14921		
6	-1	11	2.97605	3.15786	3.13766	3.14343	3.13997	3.14459	
7	1	13	3.28374	3.12989	3.14388	3.14077	3.14210	3.14104	3.14281
8	-1	15	3.01707	3.15041	3.14015	3.14201	3.14139	3.14175	3.14139
9	1	17	3.25237	3.13472	3.14256	3.14136	3.14168	3.14154	3.14164
10	-1	19	3.04184	3.14710	3.14091	3.14174	3.14155	3.14162	3.14158
11	1	21	3.23232	3.13708	3.14209	3.14150	3.14162	3.14158	3.14160
12	-1	23	3.05840	3.14536	3.14122	3.14165	3.14158	3.14160	3.14159

Students can improve their pen and compass skills by drawing a circle with a compass, increasing the spread of the compass arms and stepping around the circle drawing circles to produce a design similar to that shown in Figure 16.

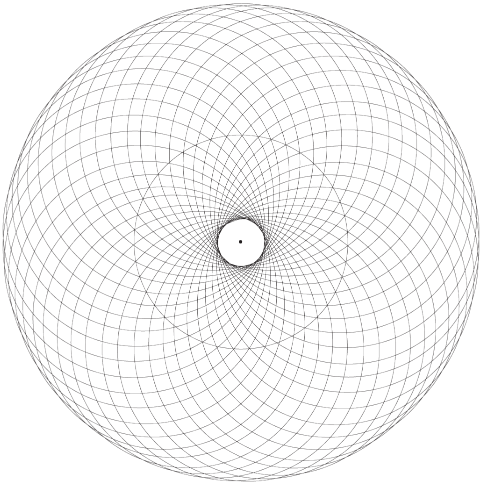


Figure 16. Pen and compass construction

After doing this students should have little difficulty in constructing the regular polygons show in Figure 17 by increasing or decreasing the compass arm spread.

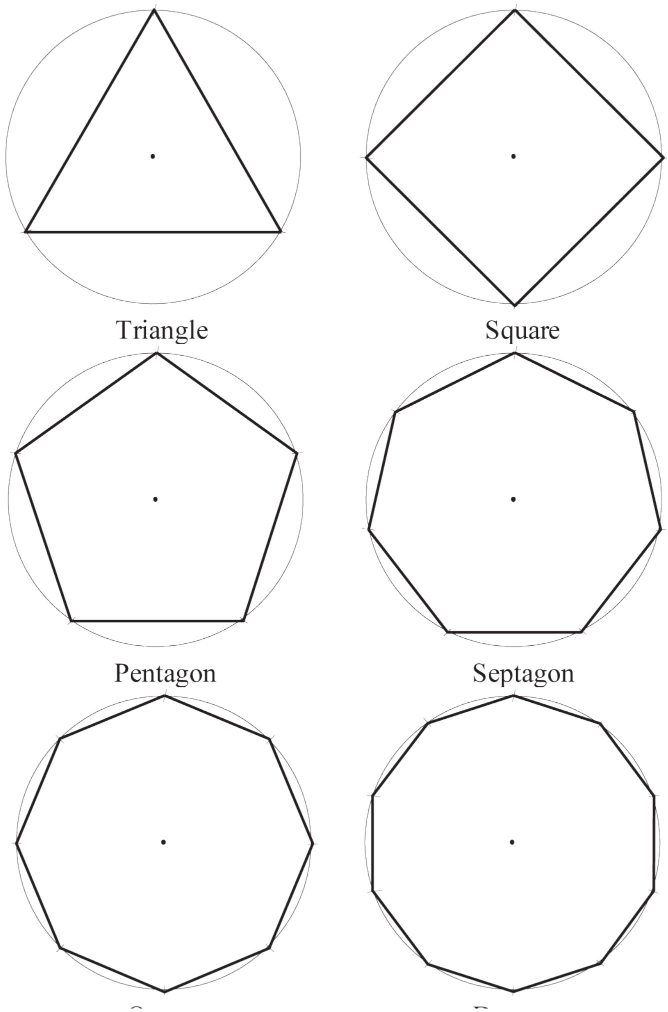


Figure 17. Regular polygons

1, 2, 4, 8, 16, ...

Dots are 'randomly' placed on a circle and connected to every other dot by a straight line so that no more than two straight lines pass though the same point. For a given number of dots, how many lines are there and what is the maximum number of regions enclosed within the circle?

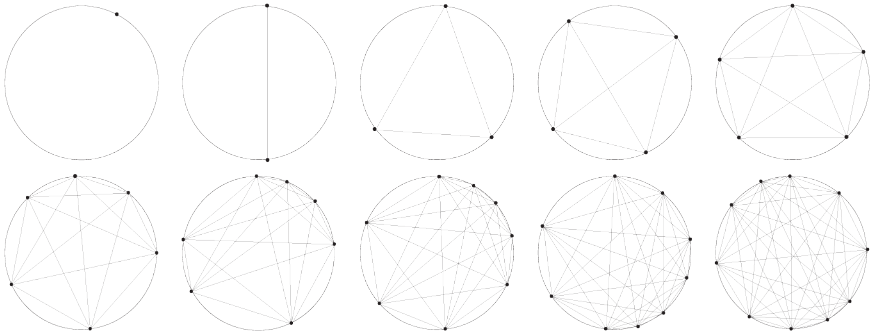


Figure 18. Dots from 1 to 10 connected to every other dot on each circle

Table 3. The circled numbers can be left off for students to complete.

Number of dots *(n)*	1	2	3	4	5	6	7	8	9	10
Number of lines *(l)*	0	1	3	6	10	15	21	28	36	45
Maximum regions *(r)*	1	2	4	8	16	31	57	99	163	256

The number of lines is given by $l = \dfrac{n(n-1)}{2}$ (these are triangular numbers).

What initially appears to be simple doubling is much more complex with the maximum number of regions given by $r = \dfrac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}$.

When older students are given the task of filling in Table 3, some leave it to the last moment writing 32, 64, 128, 256 in the blank spaces where the circled numbers are for the maximum number of regions. They usually change 256 to 512 assuming a mistake has been made in the tenth maximum number of regions. Most are able to correctly find the number of lines and hence only score 5 or 6 out of 10.

Students usually find that drawing a diagram is usually the quickest and easiest method for a small number of dots (less than 7).

The leftmost column in Table 4 gives the maximum number of regions, whilst, the second last column gives the number of dots. Each row is produced by adding the two numbers immediately above (similar to Pascals Triangle). This can be easily implemented on a spreadsheet shown on the right half of Table 3. Cells A1 and B1 are filled with the number, highlighted and filled across to cell E1. Cell A2 is filled with the formula =A1+B1, highlighted and filled across to cell E2. Row 2 is highlighted from A2 to E2 then filled down.

Table 4. A numerical method for evaluating the maximum number of regions.

1 1 1 1 1
2 2 2 2 1
4 4 4 3 1
8 8 7 4 1
16 15 11 5 1
31 26 16 6 1
57 42 22 7 1
99 64 29 8 1
163 93 37 9 1
256 130 46 10 1

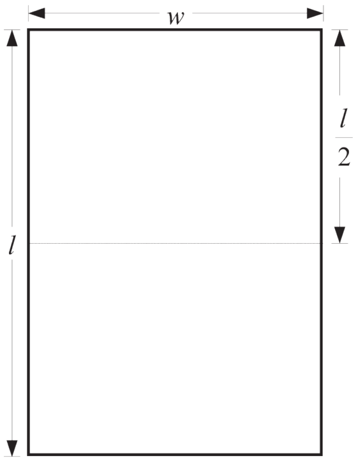
	A	B	C	D	E	F
1	1	1	1	1	1	
2	2	2	2	2	1	
3	4	4	4	3	1	
4	8	8	7	4	1	
5	16	15	11	5	1	
6	31	26	16	6	1	
7	57	42	22	7	1	
8	99	64	29	8	1	
9	163	93	37	9	1	
10	256	130	46	10	1	
11						

If the number of dots is large then many additions are necessary. Less computation is required in this case using the formula. For many students this provides their first appreciation of the 'power' of algebra.

Algebra

Asking the question, *"How far is it around the earth?"* followed by the comment, *"You stand on it every day"*, usually elicits a wide range of responses. This provides the opportunity to explain that the metre was defined a one ten millionth of the distance from the equator to the North Pole, so a quarter of the way around the earth is 10 000 000 metres or 10 000 kilometres. Hence 40 000km is the distance around the earth (assuming great circle circumference). Subsequently, holding up a sheet of A4 paper and asking, *"What has this to do with the size of the earth?"* may be answered by a different answer than, *"Nothing"*. To enlarge or reduce a page on a photocopier generally requires the photocopy paper to be the same shape as the original page. What are the dimensions of a sheet of paper of area 1m^2 which when folded in half along its long side is the same shape?

Let the length and width of the sheet of paper be l and w respectively



$$\frac{l}{w} = \frac{w}{\left(\frac{l}{2}\right)}$$

$$\frac{l^2}{2} = w^2$$

$$l^2 = 2w^2$$

$$l = \sqrt{2}w$$

Now $l \times w = 1$

i.e. $\sqrt{2}w^2 = 1$

$$w = \sqrt{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt[4]{2}}$$

$$l = \sqrt{2}\sqrt{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{\frac{2}{\sqrt{2}}}$$

$$l = \sqrt{\sqrt{2}}$$

$$= \sqrt[4]{2}$$

Figure 19. Application of algebra to paper shape

The dimensions are approximately 1.189 m × 0.841 m

This size is known as A0.

If the A0 sheet is cut in half along its long side, the 2 sheets size is known as A1.

If the A1 sheets are cut in half along their long sides, the 4 sheets size is A2.

If the A2 sheets are cut in half along their long sides, the 8 sheets size is A3.

If the A3 sheets are cut in half along their long sides, the 16 sheets size is A4.

The dimensions of an A4 sheet is 297mm × 210mm

The B series has the same shape as the A series, the longer side of B0 being $\sqrt{2}$m.

i.e. B0 has dimensions $\sqrt{2}\text{m} \times 1\text{m}$ (approximately 1.414m × 1.000m). Similarly C0 has dimensions $\sqrt{\sqrt{\sqrt{2^3}}}\,\text{m} \times \sqrt{\sqrt{\sqrt{\tfrac{1}{2}}}}\,\text{m}$ or $2^{\frac{3}{8}}\text{m} \times 2^{-\frac{1}{8}}\text{m}$ (approx. 1.297m × 0.917m) (approx. 1.297m × 0.917m).

Proofs, patterns and definitions

The following algebraic 'proofs' show the importance of the order of operations, division by zero and introduction of extraneous results by squaring and the need to consider both positive and negative square roots.

Proof that a full time worker doesn't work at all

Assume 366 days per year. Full time worker only works 8 hours a day i.e. a third of a day. One third of 366 is 122 days, so a full time worker only works 122 days less weekends and holidays. There are 52 weekends in most years so a full time worker only works 122 days, less

104 days which is 18 days. The full time worker has at least 3 weeks annual holidays. 3 weeks at 5 days per week (as weekends have earlier been accounted for) is 15 days, so a full time worker only works for 3 days (18 days less 15 days). But a full time worker has Good Friday, Christmas Day and Boxing Day off so full time worker works 3 days less 3 days which is zero days!

Algebraic Proof that 2 = 1

$$\begin{array}{lrl}
 & \text{let } a = b & \text{where } a, b \neq 0 \\
 & a^2 = ab & \text{multiply by } a \\
 & a^2 - b^2 = ab - b^2 & \text{take } b^2 \\
 & (a-b)(a+b) = b(a-b) & \text{factorise} \\
 & a + b = b & \text{cancel common factor } (a-b) \\
 & a + a = a & \text{replace } b \text{ with } a \text{ since } a = b \\
\text{so} & 2a = a & \\
\text{and} & 2 = 1 & \text{divide by } a\ (a \neq 0)
\end{array}$$

Arithmetic Proof that 2 = 1

$$\begin{array}{rl}
2 \times 0 = 1 \times 0 & \\
2 = 1 & \text{cancel common factor } 0
\end{array}$$

Proof that an Elephant weighs the same as an Ant

Denote the mass of the ant and elephant by a and e respectively and the sum of their masses by $2w$

Then $a + e = 2w$

so $e = \text{-}a + 2w \quad \ldots(1)$

and $e - 2w = \text{-}a \quad \ldots(2)$

Multiplying together left and right sides of (1) and (2)

gives $e(e - 2w) = \text{-}a(\text{-}a + 2w)$

i.e. $e^2 - 2ew = a^2 - 2aw$

Adding w^2 to both sides

gives $e^2 - 2ew + w^2 = a^2 - 2aw + w^2$

$$(e - w)^2 = (a - w)^2$$

so $e - w = a - w$

Adding w to both sides

$$e = a$$

Hence an elephants weigh the same as an ant!

Indices

Despite knowing that $2^3 = 8$, $2^2 = 4$ and $2^1 = 2$ most students are inclined to answer that $2^0 = 0$. This is understandable for those who think that 23 = 6 and this misunderstanding can be overcome by showing that $2^3 = 2 \times 2 \times 2$ but 2^0 is not so easily understood. Additionally students are inclined to write $2^{-1} = -2$. The patterns show here help students see that $2^0 = 1$, $2^{-1} = \frac{1}{2}$, etc.

$$
\begin{array}{llrlllrlllr}
2^4 & = & 2\times2\times2\times2 = 16 & & 3^4 & = & 3\times3\times3\times3 = 81 & & 10^4 & = & 10\times10\times10\times10 = 10\,000 \\
2^3 & = & 2\times2\times2 = 8 & & 3^3 & = & 3\times3\times3 = 27 & & 10^3 & = & 10\times10\times10 = 1000 \\
2^2 & = & 2\times2 = 4 & & 3^2 & = & 3\times3 = 9 & & 10^2 & = & 10\times10 = 100 \\
2^1 & = & 2 = 2 & & 3^1 & = & 3 = 3 & & 10^1 & = & 10 = 10 \\
2^0 & = & ? = 1 & & 3^0 & = & ? = 1 & & 10^0 & = & ? = 1 \\
2^{-1} & = & ? = \frac{1}{2} & & 3^{-1} & = & ? = \frac{1}{3} & & 10^{-1} & = & ? = \frac{1}{10} \\
2^{-2} & = & ? = \frac{1}{4} & & 3^{-2} & = & ? = \frac{1}{9} & & 10^{-2} & = & ? = \frac{1}{100} \\
2^{-3} & = & ? = \frac{1}{8} & & 3^{-3} & = & ? = \frac{1}{27} & & 10^{-3} & = & ? = \frac{1}{1000} \\
2^{-4} & = & ? = \frac{1}{16} & & 3^{-4} & = & ? = \frac{1}{81} & & 10^{-4} & = & ? = \frac{1}{10\,000}
\end{array}
$$

Writing ten raised to negative powers as decimals with zero preceding the decimal point, makes it straightforward for students to remember how to write the basic numeral as the number of zeroes matches the index e.g. $10^{-3} = 0.001$ which has 3 zeroes just as $10^3 = 1000$ also has 3 zeroes.

Prime Number Pattern

The following method appears to be a novel and predictable way to produce successive prime numbers. Starting with 1 and 3, obtain successive terms by adding the two preceding numbers together. This forms the Lucas Sequence. Take one from each number in the Sequence. If the term number is a factor of this number, circle the term number as shown in the table below. The circled numbers are all prime numbers. This method gives all the 126 prime numbers up to and including 701.

Table 4. The first 6 prime numbers obtained and 14 terms evaluated

Term Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Lucas Sequence	1	3	4	7	11	18	29	47	76	123	199	322	521	843
Less One	0	2	3	6	10	17	28	46	75	122	198	321	520	842

The 705th term of the Lucas Sequence less one is 2 169 133 972 532 938 006 110 694 904 080 729 167 368 737 086 736 963 884 235 248 637 362 562 310 877 666 927 155 150 078 519 441 454 973 795 318 130 267 004 238 028 943 442 676 926 535 761 270 635 which is divisible by 705. 705 is clearly not prime. This remains the longest simple 'pattern' we know of, which 'fails'.

What does $0 \div 0$ equal?

Since $0 \div a = 0$ for all real values of a other than $a = 0$ it would seem 'reasonable' that $0 \div 0 = 0$.

Since $a \div a = 1$ for all real values of a other than $a = 0$ it would seem 'reasonable' that $0 \div 0 = 1$

In a sequence where each term is the sum of the two preceding terms, where a is the first term and b is the second term, then for all real values of a and b other than $a = 0$, $b = 0$ the ratio of each term to the preceding term approaches the Golden Ratio $\left(\frac{1+\sqrt{5}}{2}\right)$ so it would seem 'reasonable' that $0 \div 0 = \left(\frac{1+\sqrt{5}}{2}\right)$.

The ratio of the circumference to the diameter of any circle is pi. As the circle becomes smaller and smaller, the circumference and diameter both approach zero, so it would seem 'reasonable' that $0 \div 0 = \pi$

As $|a|$ approaches zero $|b \div a|$ becomes larger as for all real non-zero values of a and b. Hence it would seem 'reasonable' that $0 \div 0 = \pm\infty$.

As $x \times 0 = 0$, where x is any real number, $x = 0 \div 0$ so it would seem 'reasonable' that $0 \div 0 =$ any real number.

Clearly $0 \div 0 =$ is undefined and this provides a suitable lead-in to differential calculus.

Definitions

Asking students *"What is an angle?"* often reveals a lack of knowledge. Getting students to circle the largest angle in Figure 20 is informative. Except for the right angle, all acute angles are the same so the last (reflex) angle is largest.

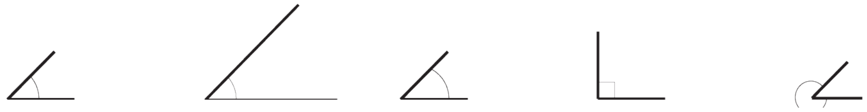


Figure 20. Circle the largest angle

When an angle is defined as a measure of how open a book (or bird's beak or hinged door) is, students realise that the size of the arms and the radius of the angle arc are inconsequential to the size of the angle, although the latter is usually larger for small angles.

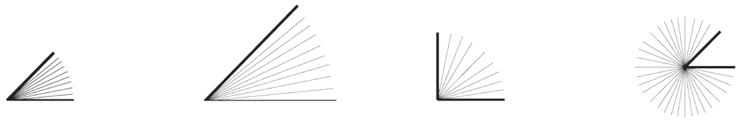


Figure 21. Which book is most open?

An accident investigator may be interested in describing how a car, which ends up facing the direction in which it was heading after a spin, got there. The car may have spun either clockwise or anti-clockwise (viewed from above). It may also have spun any number of complete turns plus a half turn. To describe this using angles requires that the angle definition be extended beyond the 'openness' of a book to encompass direction (clockwise or anti-clockwise) and angles greater in magnitude than 360°.

"Why are there 360 degrees in a circle why not 400?" is a rhetorical question worth exploring. A suitable answer is because there are about 360 days in a year (a star viewed at the same time on successive nights will have moved nearly one degree. Additional reasons are that 360 has a lot a factors and it is relatively easy to divide a circle into 6 equal pieces. There are 400 gradians in a circle and pi radii. The more mathematically useful measure is radians which can be defined as the distance around the relevant arc of a unit circle (circle with unit radius). As the distance around a semi-circle of unit radius is π then π and 180° are equivalent.

Area may be defined as the amount of surface. Getting students to cut a circle and square of equal area out of uniform material is a useful activity. Discussion of Figure 22 as to the area of the outer circle and inner circle in relation to the square, and, the area of the outer square and inner square in relation to the circle beforehand helps. This activity can be assessed by weighing each circle and square and expressing the lighter as a percentage of the larger. A significant number of students mistakenly think that shapes with the same perimeter have the same area.

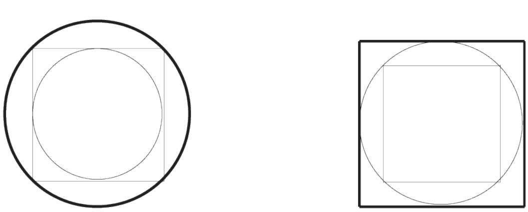


Figure 22. Squaring the circle

Teach for understanding

Directed Numbers

Rote learning 'two negatives give a positive' leads to confusion e.g. -3 – 1. By distinguishing the actions of addition and subtraction from the directions of positive and negative and using their *italic* equivalents, understanding is achieved.

Table 5. Subtraction of negative numbers using number line

Action		Direction	
add +	*step forward*	positive $^{+}$	*face right*
subtract –	*step backward*	negative $^{-}$	*face left*

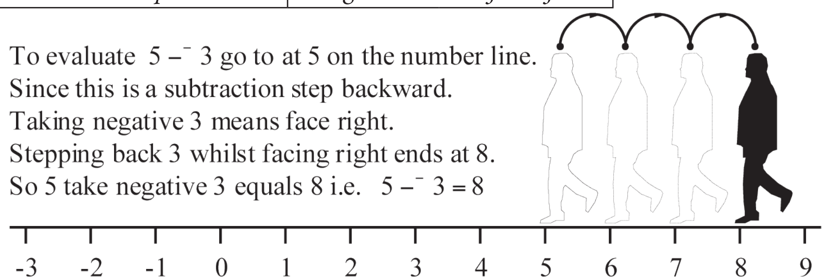


Figure 23. Subtracting negatives

Sum of Interior Angles in a Triangle

When asked for the sum of the interior angles of a triangle many students are able to reply correctly, 180°. The following often indicates whether or not the students understand this. A triangle, has been torn into three pieces (Figure 24).

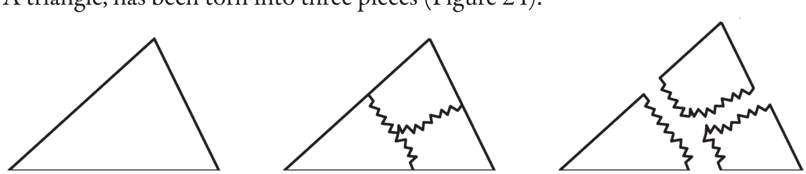


Figure 24. Interior angles of a triangle

The pieces are re-arranged with the vertices (corners) touching and the edges aligned (no overlap). Which one of the following shows how it could look or would the edges not fit together as shown but would have gaps between them?

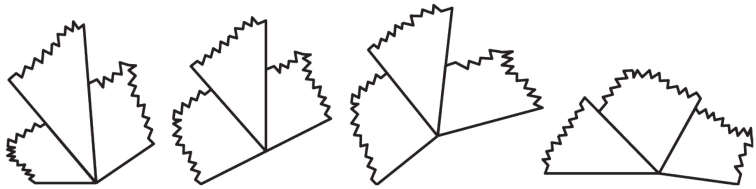


Figure 25. Combining the interior angles of a triangle

It is a simple matter to cut out any triangle with scissors and paper and tear it as in Figure 24 and combine the pieces as in the second picture in Figure 25. Some students who know that the interior angles of a triangle add to 180° fail to select the second picture indicating a lack of understanding.

Expansion and Factorisation

These operations are the inverse of one another. Each term in the first parentheses is multiplied by all the second parentheses as shown in Figure 26.

$$\begin{aligned}(2x-7)(3x+1) &= 2x(3x+1)-7(3x+1)\\ &= 6x^2+2x-21x-7\\ &= 6x^2-19x-7\end{aligned}$$

Figure 26. Expansion using the distributive law

When factorising $6x^2 - 19x - 7$, how is $+2x - 21x$ chosen to replace $-19x$? Computing the outer product i.e. $6x^2 \times -7 = -42x^2$ then listing the pairs of factors until a pair which add to $-19x$, is how. If no pair can be found, the quadratic trinomial is not able to be factorised over the integers.

e.g. $x, -42x;\ \ 2x, -21x;\ \ 3x, -14x;\ \ 6x, -7x;\ \ 7x, -6x;\ \ 14x, -3x;\ \ 21x, -2x;\ \ 42x, -x$.
Clearly, the second pair meet this requirement

$$\begin{aligned}6x^2-19x-7 &= 6x^2+2x-21x-7\\ &= 2x(3x+1)-7(3x+1)\\ &= (2x-7)(3x+1)\end{aligned}$$

Figure 27. Factorisation - reverse of distributive law

Measuring

Being able to correctly read a measure off a scale is a mathematical life skill.

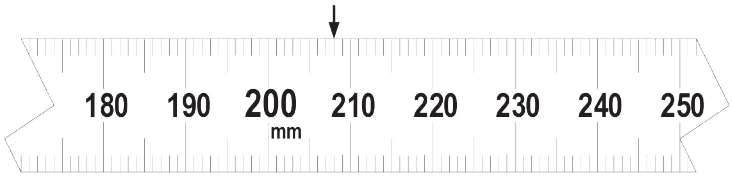


Figure 28. What reading is the arrow pointing to?

It is quite revealing to see how few final year students, keen to pursue a trade career, are unable to correctly find the measurement required in Figure 29.

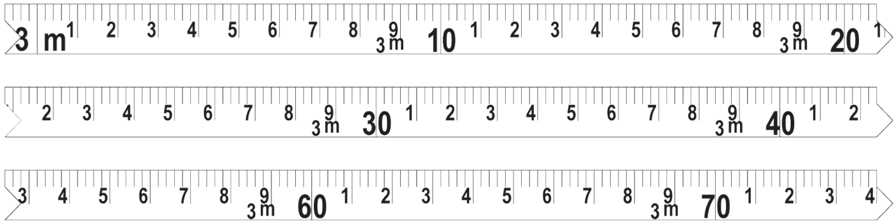


Figure 29. This tape is in three consecutive pieces. Find and mark 3·065m

Scissors and Paper

A significant number of students are 'good with their hands'. Some of these students excel in scissor and paper (cardboard) tasks. Constructing the five Platonic Solids from their nets can be a rewarding and learning experience for them. A year calendar can be made using the dodecahedron with each of the twelve months appearing on each of the twelve faces.

These solids can be used to check or establish Euler's Rule $v + f = e + 2$ where v is the number of vertices, f is the number of faces and e is the number of edges.

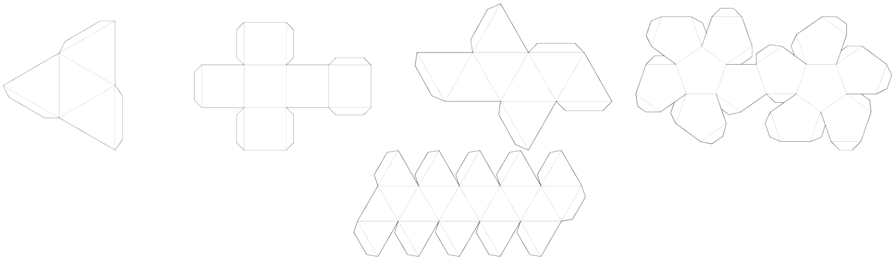


Figure 30. Platonic solid nets

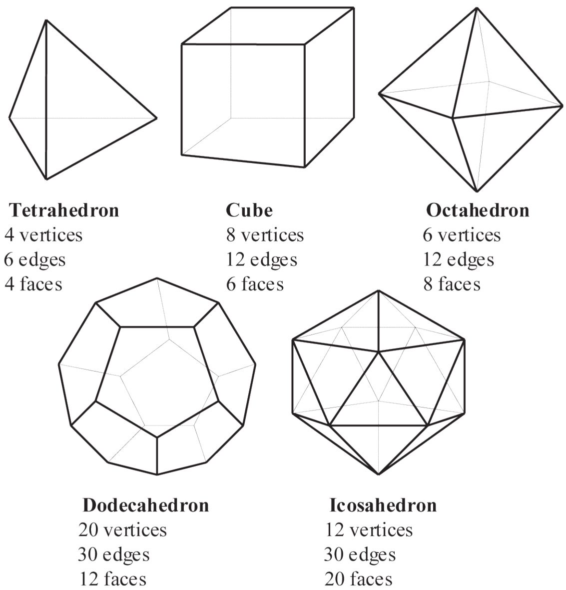


Figure 31. Platonic solids

Isometric Drawing

Many mathematically disengaged students spend much of their class time drawing. Getting them to participate in mathematics activities that involve drawing is considerably easier than getting them to engage in other mathematics activities. Isometric sketching on "dotty" paper is just one activity. This readily leads to discussions about volume, surface area as well as Euler's Rule.

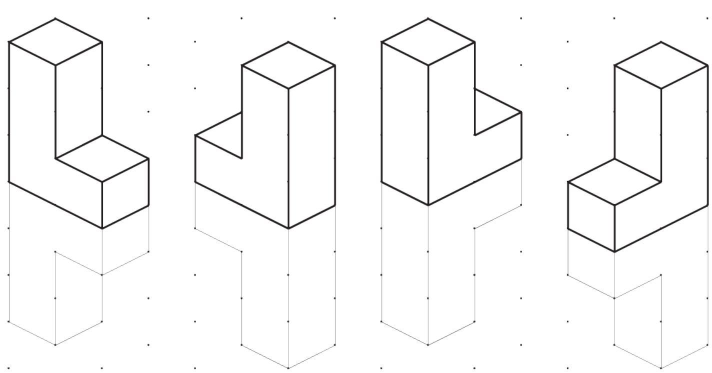


Figure 32. Isometric sketch and 3D reflection on 'dotty' paper

Point plotting

Constructing a drawing by plotting Cartesian points and connecting the dots with line segments is an activity enjoyed by junior secondary students. It is easily checked by holding students plot over a correct plot against a window.

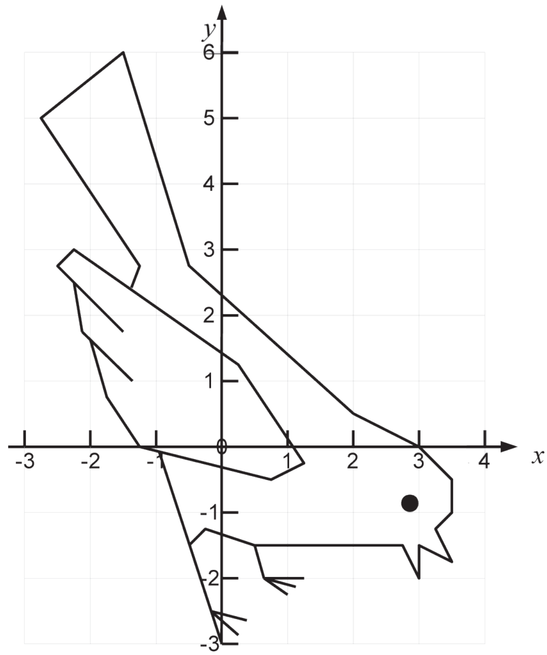


Figure 33. Connected line segments between plotted points. (adapted from Boyle 1971)

References

Beadle, P., (2011). Five top tips for beginning teachers, *Inside Teaching – the Professional Journal of the Australasian Teacher Regulatory Authorities ATRA,* 2(1), 28. (ISSN 1838-8868, Australian Council for Educational Research – ACER Press, Retrieved from http://www.vit.vic.edu.au/forms/publications/Pages/InsideTeaching.aspx)

Boyle, P. J. (1971). *Graph gallery*. Palo Alto, CA: Creative Publications Inc.

HREF1: A metaphorical "Royal Road" in famous quotations. (2012). Retrieved from http://en.wikipedia.org/wiki/Royal_Road

AN AMAZING THEOREM

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Marden's theorem gives a geometric relationship between the roots of a cubic polynomial and the roots of its derivative. It shows that the connection between a cubic polynomial and its derivative is perfectly mirrored by the connection between an ellipse and its foci. In this paper Marden's theorem is stated and illustrated with several detailed examples. A brief history of the theorem is also given.

Marden's Theorem

A cubic polynomial with complex coefficients has three roots in the complex plane and in general these roots form a triangle. The Gauss-Lucas Theorem (Needham 1998 p259, Van Vleck 1929, HREF1) says that the roots of the derivative of a cubic polynomial lie within this triangle. The result popularly known as Marden's Theorem is a sharp refinement of this. It gives an amazing geometric relationship between the roots of a cubic polynomial and those of its derivative, showing that the connection between a cubic polynomial and its derivative is perfectly mirrored by the connection between an ellipse and its foci.

In geometry, the Steiner inellipse of a triangle is the unique ellipse inscribed in the triangle and tangent to the sides at their midpoints (Minda and Phelps 2008, HREF2, HREF3). Marden's Theorem states (Kalman 2008, Kalman 2009 p42, HREF4):

Let $p(z)$ be a cubic polynomial with complex coefficients and whose roots z_1, z_2, z_3 are non-collinear points in the complex plane. Let T be the triangle with vertices at z_1, z_2, z_3. Then the roots of $p'(z)$ are the foci of the Steiner inellipse of T.

Furthermore, the root of the double derivative $p''(z)$ is located at both the centre of the Steiner inellipse of T and the centroid of T (Kalman 2008, Kalman 2009 p51, HREF4). This follows from the fact "that the average of the roots of a polynomial is the same as the average of the roots of the derivative." (Kalman 2009 p50, HREF4) and:

- The coordinates of the centroid of a triangle are the arithmetic means of the coordinates of the three vertices (HREF5).
- The centre of an ellipse lies halfway between its foci.

Investigating Marden's Theorem

To investigate Marden's Theorem it is natural to first choose three complex non-collinear points $\{z_1, z_2, z_3\}$ to be the vertices of a triangle $\Delta_{z1,\, z2,\, z3}$ and then interpolate these points with the (monic) cubic polynomial $f(z)=(z-z_1)(z-z_2)(z-z_3)$. The roots of the derivative of $f(z)$ are then found and from these foci and the midpoints of the sides of $\Delta_{z1,\, z2,\, z3}$ the equation of the inscribed ellipse can be found. Note however, that ".... if the initial three points are taken as arbitrary complex points, the expressions that arise [can] become ... large and cumbersome." (Lopez 2010).

Example 1

Consider $f(z)=z^3+3z-4,\ z\in C$. The roots of $f(z)$ are $z=1$, $z=\dfrac{-1-i\sqrt{15}}{2}$,

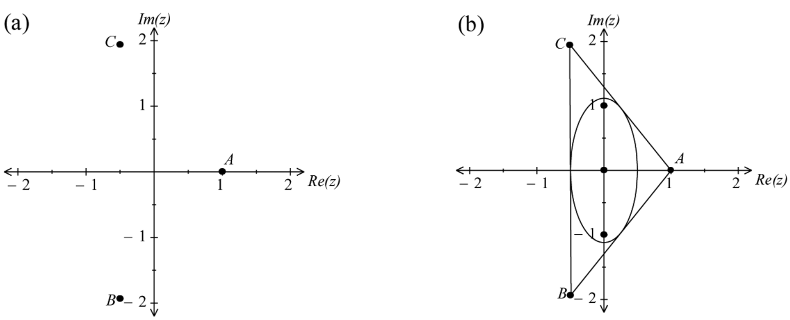

Figure 1. (a) Argand diagram showing the roots of $f(z)=z_3+3z-4$. (b) Steiner inellipse of ΔABC showing its foci at $z=i$ and $z=-i$ and centre at $z=0$.

and $z=\dfrac{-1+i\sqrt{15}}{2}$, labeled *A*, *B* and *C* respectively in Figure 1(a). Let ΔABC be the triangle with vertices at these roots (see Figure 1(b)).

$f'(z)=3z^2+3$ and so the roots of $f'(z)$ are $z=i$ and $z=-i$. These roots are the foci of the Steiner inellipse E of ΔABC (see Figure 1(b)).

$f''(z)=6z$ and so the root of $f''(z)$ is $z=0$. Note that it lies halfway between the roots of $f'(z)$ and is also the average of the three roots of $f(z)$. This root is the centre of E and the centroid of ΔABC.

Finding the Equation of the Steiner Inellipse

Marden's theorem says the following about E:

- The foci are at $z=i$ and $z=-i$.
- The centre is at $z=0$ (that is, at the point $(0,0)$).
- The midpoints of each side of ΔABC lie on E:

Midpoint of AB: $\left(\frac{1}{4}, \frac{-\sqrt{15}}{4}\right)$.

Midpoint of AC: $\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right)$.

Midpoint of BC: $\left(\frac{-1}{2}, 0\right)$.

- Each side of ΔABC is tangent to E at the midpoint. Therefore the gradient of E at the midpoints of AB and AC is equal to the gradient of AB and AC respectively (note that m_{BC} is undefined):

$$m_{AB} = \frac{\sqrt{15}}{3} \text{ and so } \frac{dy}{dx} = \frac{\sqrt{15}}{3} \text{ at } \left(\frac{1}{4}, \frac{-\sqrt{15}}{4}\right)$$

$$m_{AC} = \frac{-\sqrt{15}}{3} \text{ and so } \frac{dy}{dx} = \frac{-\sqrt{15}}{3} \text{ at } \left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right).$$

Cartesian equation

Since the foci of E are at $z=i$ and $z=-i$ it follows that the major axis of E is parallel to the imaginary axis. Therefore E is not rotated and so its equation can be found using either of the following two models:

- $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ .... (1)
- $cx^2 + dy^2 + ex + fy = 1$.

Substituting the centre $(0,0)$ into equation (1) gives the refined model

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{.... (2)}$$

Substituting $\left(\frac{-1}{2}, 0\right)$ into equation (2) and re-arranging gives $a^2 = \frac{1}{4}$. Substituting either of the other midpoints into equation (2) and re-arranging gives $\frac{1}{a^2}+\frac{15}{b^2}=16$ from which it follows that $b^2=\frac{5}{4}$. Therefore the cartesian equation of ***E*** is

$$4x^2+\frac{4y^2}{5}=1 \qquad \text{.... (3)}$$

Implicitly differentiating equation (3) gives $8x+\frac{8y}{5}\frac{dy}{dx}=0$ from which it is readily verified that ***E*** is tangent to ***ΔABC*** at the midpoint of each side.

Complex relation

The complex relation

$$|z-z_1|+|z-z_2|=k, \quad |z_2-z_1|<k \qquad \text{.... (4)}$$

defines an ellipse with foci at $z=z_1$ and $z=z_2$ and a major axis of length equal to ***k*** (Kermond 2004).

Substituting the foci at ***z=i*** and ***z=-i*** into equation (4) gives the refined model

$$|z-i|+|z+i|=k. \qquad \text{.... (5)}$$

Substituting the midpoint $z=\frac{-1}{2}$ of ***BC*** into equation (5) gives $\left|\frac{-1}{2}-i\right|+\left|\frac{-1}{2}+i\right|=k$ implying $k=\sqrt{5}$.

Therefore ***E*** is defined by the complex relation

$$|z-i|+|z+i|=\sqrt{5}. \qquad \text{.... (6)}$$

It is readily verified that equation (6) reduces to equation (3) (subject to the redundant restriction $y>-\frac{5}{4}$) by substituting ***z=x+iy*** and simplifying. This confirms that the foci of equation (3) are indeed the roots of ***f'(z)***.

Example 2

Consider $f(z)=z^3+i,\ z\in C$. The roots of *f(z)* are $z=i$, $z=\frac{-\sqrt{3}-i}{2}$ and $z=\frac{\sqrt{3}-i}{2}$, labeled ***A***, ***B*** and ***C*** respectively in Figure 2(a).

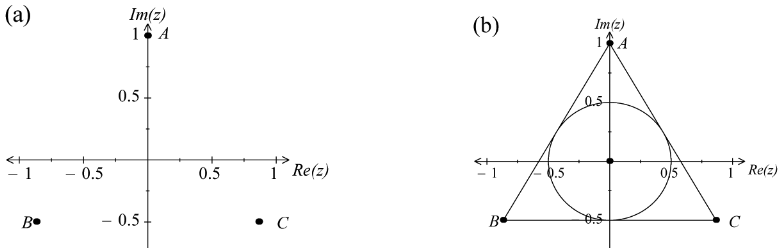

Figure 2. (a) Argand diagram showing the roots of ***$f(z)=z^3+i$*** *. (b) The Steiner inellipse of* ***ΔABC*** *is a circle with foci coinciding at* ***$z=0$*** *.*

These roots can also be found using de Moivre's Theorem since they are the cube roots of ***$-i$***.

$f'(z)= 3z^2$ and so the (repeated) roots of ***$f'(z)$*** are ***$z=0$*** and ***$z=0$***.

$f''(z)=6z$ and so the root of ***$f''(z)$*** is ***$z=0$***.

Finding the Equation of the Steiner Inellipse

Since the foci coincide (at ***$z=0$***), the Steiner inellipse ***E*** of ***ΔABC*** is a circle (see Figure 2(b)), which is the degenerate case of an ellipse. This also follows from symmetry since ***ΔABC*** is equilateral.

The complex relation defining a circle with centre at ***$z=z_1$*** and radius ***r*** is

$|z - z_1| = r.$ .... (7)

Substituting the centre ***$z=0$*** into equation (7) gives the refined model

$|z| = r.$ .... (8)

Substituting the midpoint $z = \dfrac{-i}{2}$ of ***BC*** into equation (8) gives $r = \left|\dfrac{-i}{2}\right| = \dfrac{1}{2}$.

Therefore ***E*** is defined by the complex relation $|z| = \dfrac{1}{2}$

implying $x^2 + y^2 = \dfrac{1}{4}$.

Example 3

Consider $f(z) = z^3 - (11+13i)z^2 - (16-98i)z + 138 - 134i,\ z \in C$. It can be confirmed that ***$z=3+i$*** is a root. Therefore ***$z-3-1$*** is a linear factor and from polynomial long division it is found that ***$z^2-(8+12i)z-28+54i$*** is a quadratic factor. It follows from the quadratic formula that the roots of $z^2 - (8+12i)z - 28 + 54i$ are $z = 4 + 6i \pm \sqrt{8-6i}$.

There are two different ways of finding $\sqrt{8-6i}$ in rectangular form:

Option 1: Let $u + iv = (8-6i)^{1/2}$, where $u, v \in R$. Then ***$(u+iv)^2=8-6i$***

implying ***(u2-v2)+2uvi = 8-6i***

implying u^2-$v^2 = 8$ and uv = -3

implying $u = 3$ and $v = -1$ or $u = -3$ and $v = 1$.

Option 2: Let $rcis(\theta) = (8 - 6i)^{1/2}$. Then

$r^2 cis(2\theta) = 8 - 6i = 10cis(\alpha + 2m\pi)$ where $\alpha = \tan^{-1}\left(\frac{-3}{4}\right)$ and $m \in Z$.

Therefore $r = \sqrt{10}$ and $\theta = \frac{\alpha}{2} + m\pi$

implying $(8 - 6i)^{1/2} = \sqrt{10}cis\left(\frac{\alpha}{2} + m\pi\right)$.

Substituting $m=0$ and $m=1$ gives two distinct values:

$(8 - 6i)^{1/2} = \sqrt{10}cis\left(\frac{\alpha}{2}\right)$ or $\sqrt{10}cis\left(\frac{\alpha}{2} + \pi\right)$ where $\alpha = \tan^{-1}\left(\frac{-3}{4}\right)$.

It follows from $\alpha = \tan^{-1}\left(\frac{-3}{4}\right)$ that $\cos(\alpha) = \frac{4}{5}$. Using the double angle formula

$\cos(\alpha) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1 = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$ it therefore follows that

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{\frac{4}{5}+1}{2}} = \frac{3}{\sqrt{10}} \text{ and } \sin\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1-\frac{4}{5}}{2}} = \frac{-1}{\sqrt{10}}$$

implying $(8 - 6i)^{1/2} = 3 - i$ or $-(3 - i)$.

Therefore $z = 4 + 6i \pm (3 - i) = 7 + 5i$ or $1+7i$ and so the roots of $f(z)$ are $z=3+i$, $z=7+5i$ and $z=1+7i$, labeled ***A***, ***B*** and ***C*** respectively in Figure 3(a). Let be the triangle with vertices at these roots (see Figure 3(b)).

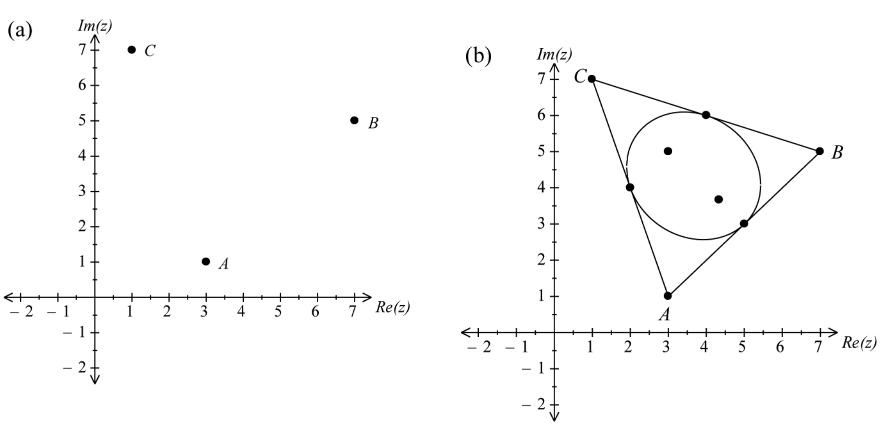

Figure 3. (a) Argand diagram showing the roots of $f(z)=z^3 - (11+13i)z^2 + 138 - 134i$. *(b)*

Steiner inellipse of ΔABC showing its foci at $z = \frac{13}{3} + \frac{11}{3}i$ *and* $z=3+5i$.

$f'(z)=3z2 - 2(11+13i)z - 16 = 98i$ and so the roots of $f(z)$ are $z = \frac{13}{3} + \frac{11}{3}i$ and $z=3 + 5i$. These roots are the foci of the Steiner inellipse E of ΔABC (see Figure 3(b)).

$f''(z) = 6z-2(11+13i)$ and so the root of $f''(z)$ is $z = \frac{11+13i}{3}$. Note that it lies halfway between the roots of $f'(z)$ and is also the average of the three roots of $f(z)$. This root is the centre of E and the centroid of ΔABC.

Finding the Equation of the Steiner Inellipse

Cartesian equation

Since the foci of E are located at $z = \frac{13}{3} + \frac{11}{3}i$ and $z = 3 + 5i$ it follows that the major axis of E lies on a diagonal line. Therefore E is rotated relative to the real and imaginary axes and so its equation has the general form

$ax^2 + by^2 + cxy + dx + ey = 1$ .... (9)

where the xy-term causes the rotation (Salas and Hille 1978 p391).

Substituting the midpoints of each side of ΔABC into equation (9) gives:

- Midpoint $(5, 3)$ of AB:

$25a + 9b + 15c + 5d + 3e = 1$ .... (10a)

- Midpoint $(2, 4)$ of AC:

$4a + 16b + 8c + 2d + 4e=1$ .... (10b)

- Midpoint $(4, 6)$ of BC:

$16a + 36b + 24c + 4d + 6e = 1$ .... (10c)

Implicitly differentiating equation (9) gives

$2ax + 2by\frac{dy}{dx} + cy + cx\frac{dy}{dx} + d + e\frac{dy}{dx} = 0$ .... (11)

Substituting the midpoints and gradients of each side of ΔABC into equation (11) gives:

- Midpoint $(5, 3)$ and gradient $m_{AB} = 1$:

$10a + 6b + 8c + d + e = 0$ .... (12a)

- Midpoint $(2, 4)$ and gradient $m_{AC} = 1$:

$4a - 24b - 2c -+ d - 3e = 0$ .... (12b)

- Midpoint *(4, 6) and gradient $m_{BC} = \frac{-1}{3}$:

$24a - 12b + 14c + 3d - e = 0$ .... (12c)

Solving any five of equations (10a), (10b), (10c), (12a), (12b) and (12c) simultaneously gives $a = \frac{-7}{236}$, $b = \frac{-7}{236}$, $c = \frac{-1}{118}$, $d = \frac{15}{59}$, $e = \frac{17}{59}$.

Substituting these values into equation (9) and re-arranging gives

$$24a - 12b + 14c + 3d - e = 0 \qquad \text{.... (13)}$$

A CAS calculator can be used to draw the graph of ***E*** if equation (13) is treated as a quadratic equation in y and re-written in the form

$$y = \frac{-x + 34 \pm 4\sqrt{-3x^2 + 22x - 31}}{7} \qquad \text{.... (14)}$$

Implicitly differentiating equation (13) and re-arranging gives $\frac{dy}{dx} = \frac{30 - y - 7x}{14y + 2x - 68}$, from which it follows that the tangent to ***E*** is a vertical line when $7y + x - 34 = 0$. The coordinates on ***E*** at which the tangent is a vertical line can be found by solving $7y + x - 34 = 0$ and equation (13) simultaneously:

$$\left(\frac{11+2\sqrt{7}}{3}, \frac{91-2\sqrt{7}}{21}\right) \text{ and } \left(\frac{11-2\sqrt{7}}{3}, \frac{91+2\sqrt{7}}{21}\right).$$

It follows that the domain of ***E*** is $\frac{11-2\sqrt{7}}{3} \le x \le \frac{11+2\sqrt{7}}{3}$. This domain can be confirmed by solving $-3x^2 + 22x - 31 \ge 0$, where $-3x^2 + 22x - 31$ is the argument of the square root appearing in equation (14).

The major axis of ***E*** lies on the diagonal line $y = -x + 8$. This line is found by noting that it must pass through the foci $\frac{13+11i}{3}$ and $3 + 5i$ and shows that ***E*** is rotated by 45^0 relative to the real and imaginary axes.

The coordinates of the vertices of ***E*** can be found by solving $y = -x + 8$ and equation (13) simultaneously: $\left(\frac{7}{3}, \frac{17}{3}\right)$ and $(5, 3)$.

Complex relation

Substituting the foci at $z = \frac{13}{3} + \frac{11}{3}i$ and $z = 3 + 5i$ into equation (4) gives the refined model

$$|z - (3+5i)| + \left|z - \left(\frac{13+11i}{3}\right)\right| = k \qquad \text{.... (15)}$$

Substituting the midpoint $z = 2 + 4i$ of AC into equation (15) gives

$$|2 + 4i - (3+5i)| + \left|2 + 4i - \left(\frac{13+11i}{3}\right)\right| = k$$

implying $k = \frac{8\sqrt{2}}{3}$.

Therefore E is defined by the complex relation

$$| z - (3 + 5i) | + \left| z - \left(\frac{13 + 11i}{3} \right) \right| = \frac{8\sqrt{2}}{3} \quad .... (16)$$

It can be verified that equation (16) reduces to equation (13) (subject to the redundant restriction $3y - 3x + 14 > 0$) by substituting $z = x + iy$ and simplifying. This confirms that the foci of equation (13) are indeed the roots of $f'(z)$.

Brief History

The result known as Marden's Theorem has been widely popularised by Dan Kalman (Kalman 2008, Kalman 2009 p42, HREF4), who named the theorem after Morris Marden because he first read it in the 1966 book Geometry and Polynomials by Marden (Marden 1966). Marden attributes the theorem to Jörg Siebeck, citing a paper from 1864 that contains a more general theorem:

The zeros of the function $f(z) = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3}$ $(m_1, m_2, m_3, > 0)$ *where* z_1. z_2, z_3 *are distinct non-collinear points in the complex plane are the foci of the ellipse which touches the line segments* $[z_2,z_3],[z_3,z_1],[z_1,z_2]$ *at those three points that divide these line segments in ratios* $\frac{m_2}{m_3}, \frac{m_3}{m_1}, \frac{m_1}{m_2}$ *respectively.*

Marden's Theorem is a simple corollary: If $p(z) = (z - z_1)(z - z_2)(z-z_3)$ then

and so, with $m^1 = m^2 = m^3 = 1$, the ellipse furnished by Siebeck's Theorem is the Steiner inellipse.

Various versions of Marden's Theorem, such as the Bôcher-Grace Theorem (Clifford and Lachance 2009), have appeared since 1864 (Marden 1966). In 1920 Ben Linfield gave a statement of the theorem in a more general form (Linfield 1920):

Let p be a complex polynomial of the form $p(z) = (z - z_1)^i(z - z_2)^j(z-z_3)^k$ *whose degree* $i + j + k$ *may be larger than 3. Let* T *be the triangle formed by the roots of* p. *Then the roots of occur at the multiple roots of* p *and at the foci of an ellipse whose points of tangency to* T *divide its sides in the ratios* $\frac{i}{j}, \frac{j}{k}, \frac{k}{i}$.

The Bôcher-Grace Theorem has been recently generalised to polygons (Clifford and Lachance 2009):

Let p *be a complex polynomial of degree n and let its critical points take the form* $\alpha + \beta \cos\left(\frac{k\pi}{n} \right)$

where $k = 1, 2, \ldots n - 1$, $\alpha, \beta \in C$ *and* $\beta \neq 0$. *Then there is an inscribed ellipse interpolating the midpoints of the convex n-gon formed by the roots of* p *and the foci of this ellipse are the two most extreme critical points of* p: $\alpha \pm \beta \cos\left(\frac{\pi}{n}\right)$.

References

Clifford, J., & Lachance, M. (2009). *A generalization of the Bôcher-Grace Theorem*. Retrieved from http://arxiv.org/pdf/0910.2446.pdf

Kalman, D. (2008). *An elementary proof of Marden's Theorem.* American Mathematical Monthly, 115(4), 330-338. Retrieved from http://www1.math.american.edu/People/kalman/pdffiles/mardenAMM.pdf

Kalman, D. (2009). *Uncommon mathematical excursions: Polynomia and related realms.* Mathematical Association of America (Dolciani Mathematical Expositions #35).

Kermond, J. (2004). *Subsets of the complex plane - Algebraic and geometric approaches.* In B. Tadich et al (Eds.), Towards excellence in mathematics. Proceedings of the Mathematical Association of Victoria 41st Annual Conference (pp. 247-258). Melbourne, Australia: Mathematical Association of Victoria.

Linfield, B. Z. (1920). *On certain polar curves with their application to the location of roots of derivatives of a rational function, Transactions of the American Mathematical Society,* 27, 17-21. Retrieved from http://www.ams.org/journals/tran/1923-025-02/S0002-9947-1923-1501242-0/S0002-9947-1923-1501242-0.pdf

Lopez, R. J. (2010). *Classroom tips and techniques: Maple meets Marden's Theorem.* Retrieved from http://www.maplesoft.com/applications/view.aspx?SID=99069

Marden, M. (1966). *Geometry and polynomials, 2nd edition.* American Mathematical Society (Mathematical Surveys and Monographs Number 3).

Minda, D., & Phelps, S. (2008). *Triangles, ellipses, and cubic polynomials.* The American Mathematical Monthly, 115(Oct), 679-689. Retrieved from http://www.geogebra.org/en/upload/files/english/steve_phelps/minda%20phelps.pdf

Needham, T. (1998). *Visual complex analysis, 1st edition.* Oxford: Oxford University Press.

Salas, S. L., & Hille, E. (1978). *Calculus one and several variables.* 3rd edition. New York: John Wiley and Sons.

Van Vleck, E. B. (1929). *On the location of roots of polynomials and entire functions.* Bulletin of the American Mathematical Society, 35(5), 643-683. Retrieved from http://www.ams.org/journals/bull/1929-35-05/S0002-9904-1929-04794-3/S0002-9904-1929-04794-3.pdf

HREF1: http://www.enotes.com/topic/Gauss%E2%80%93Lucas_theorem.

HREF2: http://www.proofwiki.org/wiki/Definition:Steiner_Inellipse.
HREF3: http://www.enotes.com/topic/Steiner_inellipse.
HREF4: http://www.maa.org/joma/Volume8/Kalman/index.html.
HERF5: http://en.wikipedia.org/wiki/Centroid#Of_triangle_and_tetrahedron.

DISCUSSION OF BROADER ISSUES IN TEACHING ABOUT THE MATHEMATICS OF GAMBLING: PAST, PRESENT AND FUTURE

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Teaching about gambling to secondary students in mathematics classes is fraught with difficulty: It is not in the curriculum, and left out of text books. It is opposed by parents. Introducing commercial gambling games grooms students. It has been done badly in the past. Individual teachers do it in an ad-hoc way. Probability concepts are hard to understand. Government sponsored curriculum interventions get it wrong. The Productivity Commission is against it because it has not worked... Let's work through a range of these issues: What mathematics gambling curricula have been available? How has gambling been used in class? What have been some of the public responses to proposals to teach about gambling? What restrictions do you face in class planning? Why have Government sponsored gambling teaching materials been so bad? What are mathematics teachers' social responsibilities?

Youth, Probability Knowledge and Commercial Gambling

Commercial gambling has become a major growth industry in many parts of the world. Over the past few decades there has been unprecedented expansion in gambling availability, participation and expenditure. Growth has been particularly strong in jurisdictions where electronic gambling machines (EGMs) and large urban casinos have been widely introduced, for example Canada, the United States, Australia, New Zealand and South Africa. Other countries, including the United Kingdom, are currently undergoing this rapidly expansionary phase. And of course the internet provides endless instant opportunities to gamble, increasingly on telephones. The prevalence of problem gambling among youth is around 5%, warranting our educational efforts. Earlier work (Smith, 2003) linking poor probability knowledge to illusions of control, well known as a correlate of problem gambling, made me aware, and disappointed, that the lack of understanding of probability concepts was, and is, widespread, among gamblers and non-gamblers; welfare advisors, psychologists, gambling researchers, even mathematics teachers!

Given the ubiquity of gambling opportunities, it seems obvious that students should have specific understanding of the principles by which gambling works for its owners, denuding the players of funds over time. Students need practical direct education about this, rather than ad-hoc gambling examples from individual teachers, or general statistics curricula, which have purposes other than warning about commercial gambling, so that the important lessons about gambling are not learnt. In the broader context of problem gambling policy, it is time for less biological and psychological interpretation of excessive gamblers, and a little more rationality, clearly demonstrated to them, preferably well before they take up the pokies curse. Most people don't need preventative education to cut down the risk of them becoming excessive gamblers, but many who gamble don't understand why they are going to lose. It is time for mathematics teachers to clearly end that confusion.

The call for gambling education has been growing. In submissions to the Productivity Commission inquiry into gambling in Australia even the gambling industry offered support: "A finding and a recommendation which encourages governments to work with stakeholders to develop quality responsible gambling education programs built on the best ideas ...is now necessary." Australasian Gaming Council (2009, p. 3) Recently the independent MP Andrew Wilkie, who has been a leader in seeking to curtail EGM gambling in Australia, said gambling education was more important than ever. He noted that fewer than one in ten recent study participants said they had encountered gambling education at school (The Mercury, 2012). "That is a very alarming finding and that is important evidence for some sort of educational awareness being included in the school curriculum," he said. But exactly what should be taught has been quite unclear. Past efforts to teach about the mathematics of gambling at secondary school level have been spasmodic. They have also been variously partial, piecemeal, inaccurate, cumbersome or buried in irrelevancies.

Past attempts: Good intentions, but poor mathematics

A discursive, rather than exhaustive review of some teaching efforts about gambling follows. Around the world, mathematics teachers occasionally address the topic, often on their own initiative, sometimes to use gambling examples to illustrate probability concepts, sometimes to teach directly about the nature of gambling. Looking at what has been included in curricula by mathematics educators, or advocated by people with an interest in gambling issues, should give us an understanding of the limits and inadequacies of some of those teaching programs, and set some of the parameters for the teaching we need to undertake.

Various people have published school level gambling curricula with some mathematics content. Howard Shaffer, a leading addiction theorist, from Harvard Medical School, mandated a mathematics course to teach about gambling (Hall et al, 2001). A problem with it was that it only addressed independence and randomness, and so, helps understand why you did or did not win on your last attempt, but says nothing at all about why you are going to lose overall. Later he published advice to parents about speaking with their children about gambling, urging: *"The next time you see a news report about a lottery ticket winner or a big winner at a casino, take the opportunity to talk to your kids about the reality of chance. Young people need to learn that winning is unlikely. Teaching your children about the odds will give them the tools to make better decisions when faced with peer pressure to gamble."* (Shaffer, 2008) However, what teaching about the odds actually involves remained unstated and unexplained.

Gambling Education: Some strategies for South Australian schools (Glass, 2002) reviewed available education attempts, and led to development of some school resources that had almost no mathematics content. Going into the Victorian State election in 2002, the Australian Labor Party had a policy of improving school education in relation to gambling. Their aim was to *"develop a school program and curriculum to educate young people about problem gambling and the risks associated with gambling"* (ALP, 2002). In the next four-year term, they did try to introduce curricula discussing gambling as risky behaviour, but did nothing with regard to mathematics teaching, and did little worthwhile in their following term. A policy officer inexperienced in mathematics education developed some exercises for adult literacy teaching, where the audience was very small, and not particularly afflicted with gambling problems. Limited and poor mathematics materials were also released as sections in a series of teaching books about gambling covering non-mathematical topics. (Victoria, 2006) Typical of Government educational attempts in this area, such as that in Queensland (2006), there was little appropriate mathematics, and overly large curricula.

Such curricula don't get to the point effectively, have marginally relevant material to wade through, and are too big to be widely adopted by teachers who already have busy teaching programs and crowded curricula. The Australasian Gaming Council, i.e. the industry, had a more positive appreciation of the Queensland initiative, calling it "a well-developed responsible gambling education program…available to all schools. The gambling industry, together with educators and community representatives, was consulted in the development of the program, which was distributed through the former Office of Gaming and Racing" (2009, p. 84).

The main points about mathematics knowledge as applied to gambling have not often been effectively taught. It is hardly surprising that the Productivity Commission called for a moratorium on all teaching about gambling, recommending "Given the risk of adverse outcomes, governments should not extend or renew school-based gambling education programs without first assessing the impacts of existing programs" (Productivity Commission, 2010, 9.20). Earlier Freethy (2003) had summed up the situation in Australia: "When government positions on this issue are examined, a range of policy documents refer to the need for prevention and early intervention. Unfortunately, that is all they do. Most are unable to specify what this means. Some are honest enough to say this is an area that requires urgent attention." State governments collect large revenues from gambling, and feel obliged to spend a lot appearing to do something about the problems that gambling causes, but vested interests dominate the allocation and spending of these funds. Appropriate mathematics curricula have been on offer for some years now, but have not been taken up. In Britain the Gambling Commission, Review of Research, Education and Treatment, Final report and recommendations (2008) concluded: "Research and education are under-resourced in comparison to treatment; a strategic framework is lacking in relation to the identification of 'problem-solving' research linked to … the development of priorities for preventive education programmes and initiatives. There has been little evaluation of existing education initiatives."

As in all areas of gambling research, Canadians have done some of the leading work on improved school education. A program from Saskatchewan Health (2006) offers basic mathematics teaching for gambling in its "What are the odds?" computer game, aimed at children in grades 6 to 9. It is a comprehensive resource, which includes teaching notes, clearly defined learning objectives, and student activities that are not heavily writing and reading based. Gambler's Help Southern, in Melbourne, have engaged in practical hands on teaching about independence and randomness in mock gambling situations. In Britain, Tacade (2011) have also produced good materials.

School Education Today

Australian experts on high school mathematics education, Goos, Stillman, and Vale (2007) suggest that "Even in Secondary school, students should be encouraged to work with manipulatives or concrete materials to assist them in building intuitions about probability and data. Collecting their own data and working with simulations may also help with personal beliefs about randomness that run counter to statistical principles" (p 259). Even so, in their celebrated Australian textbook *Teaching Secondary School Mathematics*, with the teaching of mathematical content divided into six areas, including a large chapter on "teaching and learning chance and data", in the whole discussion, indeed in the whole book, there is no reference to a gambling context. Indeed, when mathematics teachers have given gambling examples when teaching probability, it has often been to teach the principles of probability generally, not directed to why you will lose on commercial chance games, not directed to, what is for us here, the relevant application of probability knowledge. That is, teachers have occasionally used gambling to teach about probability theory, but have rarely used probability theory to explain gambling outcomes.

Curriculum Practices in Teaching the Mathematics of Gambling

It is worth reviewing a typical bad example of mathematics teaching around gambling. When presenting suggested teaching about gambling at a small Adults Learning Mathematics conference in Belfast in 2006, there was an excellent library containing many books on mathematics at the house of my fine hosts. This was one: *Statistics: an introduction* by Alan Graham (1994). On page 207, it read "if a horse is given odds of 5:1 this means that given six chances, it would be expected to win once and lose the other five times." There are two things wrong with this. Firstly, to describe the long term expectation as occurring over only 6 times is an ambiguous oversimplification. I guess, and you could work it out, that from six attempts the horse would be expected to win either once or twice or not at all. Winning once is not *the* expectation, it is the average expectation. More importantly, the quoted statement fails to recognise the difference between the horse's chances (the actual odds) and the *betting* odds which are being offered by the bookmaker. Why be so petty about this? Because *it is the whole crux of why people will lose their money in commercial gambling*. Before we explain that further, let us look at how the author used his horse race information for student exercises. He gratuitously introduced the students to gambling, asking them to calculate how much they would get from winning bets on each of three horses. Nevermind losing, which would have been the more likely prospect.

Below is an example of what could have been done instead, where the emphasis is not on what we might, possibly, win, but on what we will, more definitely, lose:

Table 1
Horses' Betting Odds Expressed as Fractions

	Horse 1	Horse 2	Horse 3	Horse 4	Horse 5	Horse 6	Horse 7	Horse 8
odds	9:4	3:1	5:1	7:1	10:1	12:1	14:1	16:1
fraction	4/13	1/4	1/6	1/8	1/11	1/13	1/15	1/17
decimal	.3077	.25	.1667	.125	.0909	.0769	.0667	.0588

Sum of Probabilities = 1.1427 In a mathematically fair game this would add to one, as the total probability for all possible outcomes is 1, by definition. Take the multiplicative inverse = 0.87512 Subtract from 1 - 0.87512 = 0.12488 = Bookie's margin; an expected rate of loss by bettors of 12.5% or one-eighth of their total bets.

This example from a textbook has the additional disadvantage of actually having a lower house margin than any horse racing odds that I have ever previously examined. Twenty to 30% or even higher, is a much more common margin in actual quoted bookmaker odds at any particular time. Of course the bookmaker's actual margin is likely to be a little lower as the prices move and the punter hopefully, from their point of view, bets at the best price. Furthermore, the preponderance of betting is with the short priced horses, the 'favourites', so bookmakers cannot be sure to have their various possible payouts covered in all races.

Treating the odds for each horse as though they are out by the same proportion (not an entirely valid assumption), to bring the sum of probabilities back to 1, we get for the 5:1 horse: 0.1667 x 0.87512 = 0.1459, inverse is 6.8548 i.e. an estimate of the actual odds is 5.85:1, not the 5:1 quoted by the bookmaker.

It is the difference between the betting odds, on which the bookie pays the prize, and the actual odds, which generates the profit for the bookies to stay in business. (Apart from any fraud or corruption, which there often is in gambling contexts.) In almost all cases when odds are used in society they are used in the commercial gambling sense to indicate the rate at which a prize is to be paid, they are not calculated directly from the probability of an event, i.e., odds are not used in practice as another way of expressing the probability of an event, so perhaps mathematics teachers should not use them in that way either. That is to say- *if you teach the notion of chance odds you need to immediately distinguish it from payment odds*. Remarkably the 2012 publication offered for use in schools (Victoria, 2012, p. 50) has the same misleading approach to gambling odds. In the section on 'Understanding the odds' in a worksheet for students, after asking *"What is the difference in payout between the Quinella, Trifecta and First Four?"* it reads "The dividend [$18] for Its A Wonder [a horse] indicates that its [payment/real?] odds were 19/1. This means it was only expected to win once and lose 19 times every 20 starts." Firstly 19/1 is just plainly wrong. They may mean 18/1, but as horse dividends in Australia are quoted inclusive of return of the unit bet, 17/1 is the figure they want. They have added when they should have subtracted. Let's call it a 'typo'. We all do them. But the truth is more like one in 24, when the difference between real and payment odds is taken into account! Introducing horse betting options in the classroom, then talking rubbish, this is what the State Government of Victoria is currently offering your children.

Difficulty of Probability Concepts

There are problems in teaching about gambling probabilities which any mathematics curriculum will need to take into account: Mathematics teaching doesn't always get the point across, and probability concepts may be particularly difficult to grasp. Applying concepts to new contexts can be difficult. If we don't engage students, sometimes quite concretely, they don't always get it. There is research which questions the efficacy of knowledge about probability in gambling situations. It has long been held that *general* mathematical understanding does not innoculate against common intuitive errors about gambling (Peard, 1991, Ayres & Way, 2001). More recently Canadian researchers have addressed these issues (Benhsain & Ladoucer, 2004). In 2006, Williams and Connolly questioned whether "learning about the mathematics of gambling change[s] gambling behavior?" And Australians Lambos and Delfabbro (2005) have suggested "a basic understanding of mathematics, statistics or gambling odds is unlikely to be a protective factor in problem gambling because gamblers can pick and choose which information they chose to apply when the information is applied to activities in which they have a personal interest." As the discoverer of illusions of control pointed out in 1975: Pure chance is especially difficult for people to understand and accept, as there is no contingency, no meaning and most importantly, no control (Langer, 1975). There has been too much assumption that because university students, for example, undertake courses about statistical tests, they should understand how those principles would apply in a gambling context. Teaching needs to be explicit and concrete to maximize effective communication. If we want students to understand principles applied to a particular context we should teach them in that context. Our great hope for transferable knowledge is not often fulfilled. Amongst those mathematically inclined, much of what I am suggesting we should teach (Smith, 2011) falls into the category of the 'bleeding' obvious. The point is that it is not as obvious as some may think, and to be understood, requires teaching. People will not intuit mathematical understandings which took mathematicians hundreds of years to discover. High school education seems to be the most appropriate place for the probability of gambling to be made explicit.

High school students can be shown how the difference in pricing between the chance of a winning result and a lower corresponding payout generates guaranteed loss for players over time. Once grasped, extension of the concepts to more sophisticated chance gambling games, like electronic gaming machines (pokies) is conceptually straightforward, although the detail is almost prohibitively complicated, and often unavailable.

An example of how far away governments have been from appropriate teaching responses is seen in Scotland. "Betting among schoolchildren has reached such desperate levels that Gamblers Anonymous is stepping up a major new project aimed at curbing the problem. A [Government] spokesman said it was up to schools how to deal with the issue. He added: 'It is a matter for individual teachers, who could include gambling as part of the personal, social and development strand of the curriculum'" (Blythman, 2007). Again the response was not in terms of the mathematics of gambling.

A Confused Controversy

Even the type of gambling examples we may use in schools are fraught with controversy, as we overstep the realm of social education, and impinge on parental prerogatives about children's education. However we do not need to introduce actual commercial gambling games, in order to teach about them. Dangers with teaching the mathematics of gambling include introducing gambling to students: just the fun, not the mathematics. Introducing commercial games; teaching their rules, initiates and grooms students for gambling.

A few years ago a professor in Scotland advocated teaching basic gambling mathematics and encountered considerable resistance (Denholm, 2007). "Professor Alastair Gillespie, chairman of the Scottish Mathematical Council, believes using dice and packs of cards in secondary school lessons would help pupils learn basic maths techniques such as probabilities. He also believes it would encourage more people to take up maths. However, the suggestion sparked anger from gambling support groups who claimed using it as a routine part of maths lessons would send the wrong signals to pupils and might even encourage gambling. Professor Gillespie said: "I am not advocating gambling for children but there are some classic problems in probability which are really gambling problems. Things like tossing coins and cutting cards are simple techniques which teach pupils about basic maths and I think it would catch the interest of students if we were to introduce that in schools. What you are trying to do is engage with pupils and present them with scenarios which interest them because it shows how maths can be relevant and we need to do more of that." Professor Gillespie said it was 'unlikely' that pupils would take up gambling because, by teaching them about the realities of probability, pupils would realise how unlikely it was they would win. "It would not lead to more people gambling. If anything, once people realised the odds, they would probably not buy lottery tickets or use fruit machines," he added. However, Dr Alex Crawford, chief executive of RCA Trust, Scotland's foremost gambling counselling agency, warned that linking mathematics to gambling in schools was tantamount to promoting it. "We know gambling is a high-risk problem in Scotland and making it mainstream in this way and raising interest in gambling would concern me," he said. "The potential impact is massive. ...It is all very well to want to encourage more people to take up maths but this idea does not belong in the real world." Mark Griffiths, a leading professor of gambling studies, said there was no evidence that teaching people gambling led to problem gamblers, as long as the social impact was understood. "Gambling should be on the school curriculum because it engages people with basic maths," he said. "As long as there is an understanding that in some circumstances gambling can be a problem I don't think there is anything wrong with this." Similar debates have occurred in Australia. Gambling researcher Charles Livingstone (Productivity Commission, 2010, p. 9) said that "I think education campaigns look good, they make people feel that they're doing something; whether they actually achieve anything is very doubtful, certainly in other areas of public health. I don't think an education program in schools about the dangers of gambling is likely to do anything other than to encourage risk-taking kids to have a go. That's, bluntly put, what the literature would suggest." He made no comment on the mathematics involved.

Most recently, the leading gambling reformer, the otherwise highly respected Senator Nick Xenaphon, tried to stop a specialist talk to an MAV general audience at the Melbourne Museum. In objecting to mathematical talk about sports gambling Xenaphon wrote: "In essence, by purporting to teach individuals to assess gambling odds using a 'mathematical approach', you are in turn also encouraging people to gamble."(Ross, 2012) Surely he should be more concerned about the ubiquitous advertising of gambling, including its extensive insertion into sports broadcasting? Extreme restrictions have been placed on commercial advertising of other undesirable products, like tobacco. There is a fine line between freedom of speech, and the banning of pernicious communications. It may be drawn along the boundary of commercial activity.

Opportunities to review teaching on this topic by some individual Victorian secondary mathematics teachers produced occasional examples of good teaching about the mathematics of gambling. These rare examples may be seen against the background of authorised curriculum documents: In Victoria (VCAA, 2012) they contain the dictum that students at level 4 (aged approx 15-16) should understand a key concept needed in understanding gambling expectation, viz: "Students describe and calculate probabilities using words, and fractions and decimals between 0 and 1. They calculate probabilities for chance outcomes (for example, using spinners) and use the symmetry properties of equally likely outcomes. They simulate chance events (for example, the chance that a family has three girls in a row) and *understand that experimental estimates of probabilities converge to the theoretical probability in the long run."* Perhaps more appropriately this point ought to be made in the following year when their capacity to understand it is likely to be better developed, and when they are closer to legal gambling age. As an example in class they could consider simple gambling games.

The dilemmas about what gambling to introduce in schools are solved by having a clear understanding of which mathematics explains the outcomes from gambling, and then teaching that. Gambling is then only brought in incidentally to teach about it. There is no need to use commercial games at all, and it is certainly not necessary to outline their rules and betting options. In my own recent work I've concentrated on showing a mathematics curriculum around gambling, but it is sensible that broader issues than the mathematics of expected loss would be considered by students in secondary schools. What other content might a gambling curriculum contain? Articulating the common myths and illusions may bring awareness of how common they are, so they may be more readily avoided. Of particular interest from a numerical perspective may be some of the data informing numerical literacy about social facts of gambling, i.e. data about who spends how much, economic distortions, effects on local economies and so on., as well as non-mathematical material around risk and quality of life issues, from which there is now plenty to choose. Another worthwhile teaching restriction would be to not make gambling games more exciting than other school work by doing fun things you wouldn't do in other classes. Don't give out prizes for gambling. An exception might be to provide slightly desirable small lollies to use as betting tokens. The students will lose them if they play, and so may feel some of the disappointment which is genuinely experienced in commercial gambling. What we do not need is a poorly focused jumble of activities which do not deal directly and comprehensively with the three salient mathematical facts of commercial chance gambling, viz

1. The events are random and independent,
2. There is a specific house margin on each bet.
3. The probability distribution / reversion to the mean guarantees long term loss at the expected loss rate.

Social Responsibility of Mathematics Teachers

In numeracy education, the teacher has a responsibility to pass on the most significant mathematical social understandings of the world through real world examples (Smith, 2000). Mathematics teachers have often filled the role of alerting students to basic financial gambits, so that students are not unduly penalised by inadvertent behaviours, including use of third-party ATMs, non-repayment of outstanding credit card balances, making small withdrawals from cheque accounts when the BAD tax was extant, failing to submit a tax threshold adjustment form to an employer, and suchlike. Gambling is now clearly a potential major financial risk, one that is uniquely susceptible to mathematical analysis and understanding. It is a breach of the teacher's normal duty to their students to be leaving this out of the curriculum in the way it has been. Just because most people don't have a sufficient grasp of probability concepts to actually explain or give an account of why they will lose, is not reason to avoid such teaching, if it is available. Clearer understanding of what the issues are in teaching about gambling should enable us to proceed with sensible curriculum interventions. Let's challenge the notion that being vague about the likelihoods of gambling outcomes is satisfactory for the playing public, gambling counsellors, or educators.

References

Australian Labor Party. (2002). *Policies. Victorian Branch Australian Labor Party.*

Australasian Gaming Council. (2009). *Submission to the Productivity Commission Draft Report on Gambling* December 2009.

Ayres, P, & Way, J. (2001). *The effects of instruction on likelihood misconceptions.* In M. van den Heuval-Panhuizen (Ed.), Proceedings of the 25th Psychology of Mathematics Education Conference, Utrecht, Netherlands, Vol. 2, 73-80

Benhsain, K., & Ladouceur, R. (2004). *Knowledge in statistics and erroneous perceptions in gambling.* Gambling Research, 16, 25–31.

Blythman, J. (2007, June 17). Sunday Herald, Scotland.

Consumer Affairs Victoria. (2012). *Responsible gambling*. Victoria: Author.

Denholm, A. (2007, January 4). *Simple gambling games should be used as teaching tools in Scottish schools.* The Herald, Scotland.

Department of Justice Victoria. (2006). *Problem gambling: A guide for Victorian schools* Victoria: Author.

Freethy, C. (2003). *Prevention and early intervention - problem gambling's learning edge.* National Association for Gambling Studies (Australia), Canberra.

Gambling Commission. (2008). *Review of Research, Education and Treatment,* Final report and recommendations, UK.

Glass, L. (2002). *Gambling education: Some strategies for South Australian schools* Government of South Australia.

Goos, M., Stillman, G., & Vale, C. (2007). *Teaching secondary school mathematics: research and practice for the 21st century.* Crows Nest, N.S.W: Allen & Unwin.

Graham, A. T. (1994). Statistics: an introduction. Hodder Arnold.

Hall, M. N., Vander-Bilt, J. et al. (2001). *Facing the odds: The mathematics of gambling and other risks.* Boston, MA: Harvard Medical School's Division on Addictions, Massachusetts Council on Compulsive Gambling.

Lambos, C., & Delfabbro, P. (2005). *Cognitive biases and numerical reasoning ability in problem and non-problem gamblers [online].* In G. Coman, (Ed.), Proceedings of the 15th National Association for Gambling Studies Conference. Alice Springs, N.T: NAGS.

Langer, E. J. (1975). *The illusion of control.* Journal of Personality and Social Psychology, 32(2), Aug 1975, 311-328.

Peard, R. F. (1991). *Misconceptions in probability: A comparison of gamblers and non-gamblers within the Australian social context.* Centre for Mathematics and Science Education Technical Report No:3. Brisbane: QUT.

Productivity Commission. (2010). *Gambling inquiry report,* AusInfo: Canberra.

Queensland Government. (2006). *Responsible gambling teaching resource kit*

Ross, M. (2012). *The joy of sports gambling.* MAV Lecture, Melbourne Museum 16th September, 2012.

Saskatchewan Health. (1999). *Gambling: reducing the risks — A problem gambling prevention teaching resource to support middle level health education Grades 6, 7, 8 and 9.*Shaffer, H. J. (2008). Talking with children about gambling. Washington, DC National Center for Responsible Gaming.

Smith, D. (2003). *Gambling behaviour and understanding of probability concepts among university students.* In. L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), Mathematics education research: Innovation, networking, opportunity. Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia. Geelong, Vic: MERGA.

Smith, D. (2000). *Adult numeracy teaching: An Australian focus on social contexts.* In M. Schmitt, & K. Safford-Ramus (Comps.) (2001). Adults learning mathematics-7: A conversation between researchers and practitioners. Proceedings of ALM-7, the International Conference of Adults Learning Mathematics. Medford, MA, July 6-8, 2000 p. 87-92.

Smith, D. (2011). *Graphically explaining expected loss on chance gambling.* In L. Bragg (Ed.), Maths is Multi-Dimensional. Proceedings of the Mathematical Association of Victoria 48th Annual Conference (pp. 208-218). Melbourne, Australia: MAV.

Tacade. (2012). *Just another game? Gambling education materials for young people aged 13-19.* Produced by Responsibility in Gambling Trust and Tacade. Also available at http://www.tacade.com/gambling.php.

Victorian Curriculum and Assessment Authority. (2012). Victorian Essential Learning Standards http://vels.vcaa.vic.edu.au/vels/maths.html

Williams, R. J., & Connolly, D. (2006).*Does learning about the mathematics of gambling change gambling behavior?* Psychology of Addictive Behaviors, 20(1), 62-68.

LUCKY COLOURS OF SUNSHINE: TEACHING THE MATHEMATICS OF CHANCE GAMBLING LOSS

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Lucky Colours of Sunshine is an elegantly simple gambling game, which helps reveal the salient mathematics of gambling. Collection and display of participant data, enables thorough analysis of game outcomes, and comparison with expectation over time. Further investigations are shown, using some freely available interactive displays, which help to ram home the reasons why you cannot win in the long run on commercial chance gambling like "the pokies".

Mathematics and Chance Gambling

Last year the Mathematical Association of Victoria first published an example of proposed mathematics teaching about gambling for adolescent learners (Smith, 2011). This year I reprise teaching about the suggested game, then add dynamic, that is, interactive graphs which enable display of the likelihoods of the possible game results, over extended play. Elsewhere in these proceedings I discuss the broader educational and social issues around such teaching.

Gambling has been around for a while now. Selection by lot was done in Biblical times. In the Louvre and other major historical museums you can find examples of ancient dice in use more than 2000 years ago (see Figure 1). The mathematics of gambling has been around a while now too, though nowhere near as long in any developed form, and although probability theory began its life very much about gambling, gambling is now largely left out when probability is taught in schools.

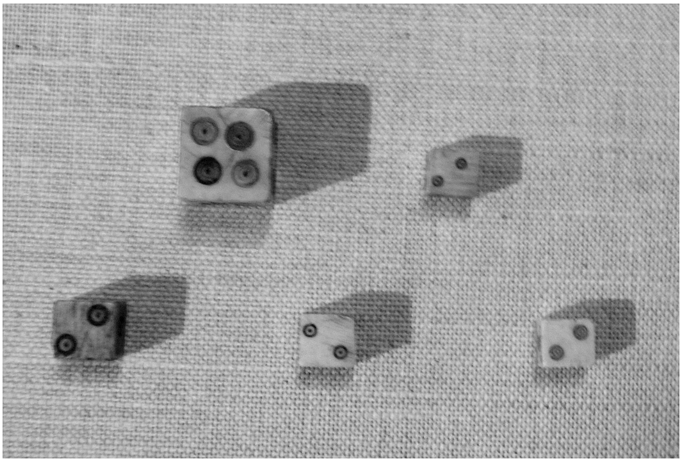

Figure 1. Ivory dice from Egypt (Roman Period)

Carefully designed teaching at a middle secondary level, can demonstrate emphatically why it is practically impossible to win over the long run on commercially available chance gambling. Mathematics fully describes the outcomes from betting on chance games. Short-term results are variable within a range of possibilities. Long-term results are less variable, with an ever-increasing certainty of long-term net loss. The results from chance gambling are a randomly patterned reflection of the structure of the game being played. Such mathematical content, can be developed with interaction from a student group, in a "hands-on" session.

Key Teaching Points

Key misunderstandings in pure chance gambling, such as on electronic gambling machines, include a lack of: understanding of independence and randomness, understanding of the basic loss making structure of the game, and appreciation of the tendency of variable short-run chance results to congregate around the average mathematically expected result in the longer run. That is, that the variability of outcomes reduces in a larger sample. So these are ideas we could try to teach. Rather than focussing directly on the myths, illusions and misunderstandings people have about gambling, we are asking what understandings they need then examining how we teach them.

The structures of commercial gambling games set up the players to lose, in that the total of prizes offered is less than the average amount of betting needed to generate those prizes. That gives a house margin, whereby the operator is ensured a profitable business. In a rather strange way, this means that each win, is actually when you 'lose', because you are not paid a prize commensurate with the unlikeliness of that win. You are typically paid a smaller prize than the frequency or rarity of the winning event would require, were it a fair game in the mathematical sense. Over time, if you keep playing, these insufficient winning payouts leave you further and further behind, as you also experience the normal range of losing bets.

In a short period of play the random distribution of results may be more or less favourable toward you, when compared with the average expected loss. It may even give you a short term win, but longer term the chance of gaining results differing markedly from the expected rate of loss diminishes practically to nothing. This patterning of results over time, increasingly reflecting the inherent game structure, has been called the law of large numbers. It is quite possible to understand this without advanced mathematics talk of normal curves, distribution theory, and the central limit theorem, which typically are only addressed by advanced mathematics students.

High school students can quickly be shown how the difference in pricing between the chance of a winning result and a lower corresponding payout generates guaranteed loss for players over time. Once grasped, extension of these concepts to much more complicated chance gambling games, like Electronic Gaming Machines (EGMs), "the pokies", is quite straightforward, although the arithmetic involved is somewhat prohibitive, and the prize frequencies sometimes unavailable as commercial in confidence.

Class Activities

I am assuming that students will receive theoretical and practical examples about probability in the classroom much as they do now, but I will argue that, in addition, we can usefully devote a few sessions directly to understanding why chance gambling produces losses for the players. We may stimulate interest in the structure of a gambling game by playing one. The game Lucky Colours of Sunshine has several features which make it useful for school teaching. It is not an existing commercial game but a simple simulation game which can help students to understand an important underpinning idea about gambling in general. It has a rate of loss consistent with common gambling products. Played in multiples of 12 it has beautifully simple arithmetic. This is a game of selecting one colour from four. If the correct colour is picked the players are paid a net prize of two units per unit bet. Players will need to record their colour choices, the colour that comes out, and the amount won or lost, an example is shown in Table 1.

Table 1

Recording Individual Results from Playing Lucky Colours of Sunshine

Game	Your choice of colour	Colour Picked out	Win or loss	Running total – number of wins	Running Total – money won or lost
1	White	Red	Loss		-1 = -1
2	Blue	Yellow	Loss		-1 → -2
3	Blue	Blue	Win	1	+2 → 0
4	Yellow	White	Loss		-1 → -1
5	Blue	Loss	Loss		-1 → -2
6					

As we play successive rounds, we may enquire as to who has won and who is still ahead. After two goes, we will very likely still have some people who are in front. Playing the game gives us the opportunity to raise various myths and illusions. If we play twelve times we will get a sufficient spread of results for instructional purposes. After playing 12 times the average expected number of wins is three, giving winnings of 9 units, and a net loss of three. Those who have won less than three times have been unlucky and have lost more than average, those who have won more than three times have been lucky, but still most of the lucky will find that they have lost overall. Doing better than average won't guarantee an overall win. Only the very lucky who have won more than 4 times will be ahead, and they will be few and likely not far ahead. Of course, it is possible that someone has won big, but such freak results are just that.

If we begin again and play another set of 12, we will get similar results, except that it is likely that different people will be amongst the very lucky, thereby demonstrating the inability of the lucky to maintain their status in a random situation. Of course, you can modify the game to introduce some form of jackpot, e.g., if the same colour comes up four times in a row, and you didn't bet on it for any of the four you can have a compensation jackpot of 10. Does that sound alluring? A maximal betting strategy is now possible. Never bet on the preceding winning colour and you can expect an increase in your payout of 10 in 64, which now reduces your expected rate of loss to about 10%.

At a more advanced level of mathematics teaching, we may predict the distribution of results from the group, and then compare the actual results. If we take our sets of results and plot or average them all, we shall see that indeed the expected results are achieved. This is quite powerful, showing that the mathematical explanation of the randomness of the outcomes concurs strongly with reality, providing a predictive model much more successful than any stratagems you may favour, such as rituals or metaphysical beliefs. The collective results of the group may be tabulated, then graphed, as has been done in Figure 2.

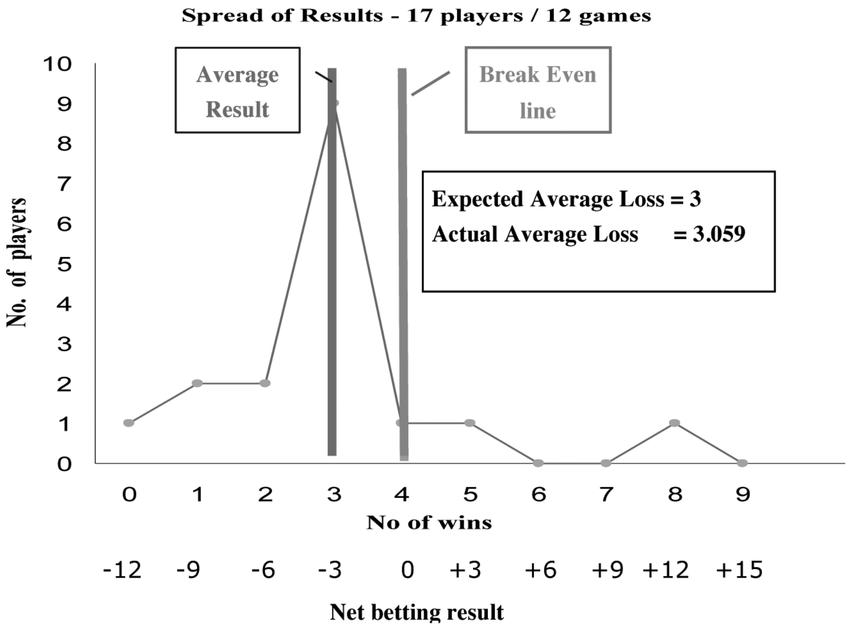

Figure 2. Actual results for 17 players after 12 goes

If we begin again and play another set of 12, we will get similar results, except that it is very likely that different people will be amongst the very lucky, thereby demonstrating the inability of the lucky to maintain their status in a random situation.

Dispelling Myths and Illusions: Questions to ask During Play

The experience of your students can be enriched by considering answers to questions you can ask during play. For example, Who is competitive when they play a game? Do you think you can improve if you try hard or practice? Do you think you can influence the colour drawn? Do you think you can do anything to become more successful in guessing the colour? If you are ahead after this set, will you be ahead on the next set?

In playing this game, people may be asked to reflect upon the intuitive illusions about gambling. They may be aware of their competitive spirit, in a game over which they can have no influence on the result. They may be aware that they are concentrating, and trying to feel the forthcoming colour. Repeated play will demonstrate that this does not work.

One becomes aware of how powerful the illusions of control are. We desire control of the situation. We are successful as humans, individually and as a species, when we have been able to recognise patterns. However, that is in situations where patterns have causal significance, and hence meaning. But, here we attribute meaning in circumstances without significance; circumstances that are random, and not just casually random, but causally random, i.e., in Australia and many other jurisdictions, by law, EGMs are designed to be random.

Analysing the Game

Consider the odds. Your chance of winning is one out of four, which is called 3 (against you) to 1 (for you). Your average expectation is one win out of four, three losses out of four. But when you win you are only paid 2 to 1. You would need to be paid at 3 to 1 on your wins to match the outlay on your expected losses. That is, you need a winning prize to equal your expected losses, in a (mathematically) fair game. This mismatch between the payment odds and the real odds is fundamental to commercial chance gambling.

We can calculate the expected average rate of loss, which I much prefer to "return to player" (RTP).

$$\textbf{RTP} = \frac{\textbf{Favourable outcomes} \times (\textbf{1+ Prize})}{\textbf{Total equally likely outcomes}}$$

$$= \frac{1(1+2)}{4} = 75\% \qquad\qquad \text{Rate of loss} = 1 - \text{RTP}$$

$$= 25\ \%\ \text{of turnover}$$

In Summary: Play the Game in Sets of 12 over Several Lessons.

Record individual results. Consider the odds of winning. Consider the average expectation of numbers of wins in a certain number of games played, and the corresponding expected loss. Graph group results, note spread, wins and losses. Do it again, a few times. Compare individual results; variation and losses. Total individual results, graph (and compare to Binomial/Normal curve). Be amazed that maths all along could tell us what distribution of results we would get from playing this random chance game. It's not essential to take up my exact suggestions, but I hope you see the possibilities.

We can go on to show the expected distribution of wins (and corresponding net betting outcomes) over increasingly large periods of play. Here this is shown in a freely available dynamic display using the maths software program Mathematica, initially available at http://ftp.physics.uwa.edu.au/pub/Mathematica/VU/Smith/ provided you first download a free CDF player in order to view it.

All the parameters are variable. Figure 3 shows an example of the output of the software program to illustrate what it does. This shows the expectations of numbers of wins from 24 games, with the average expectancy and the breakeven lines. The number of wins may be counted along the abscissa, and is scaled as a percentage. As you increase the number of games played a decreasing proportion of the graph is on the win side of the breakeven line.

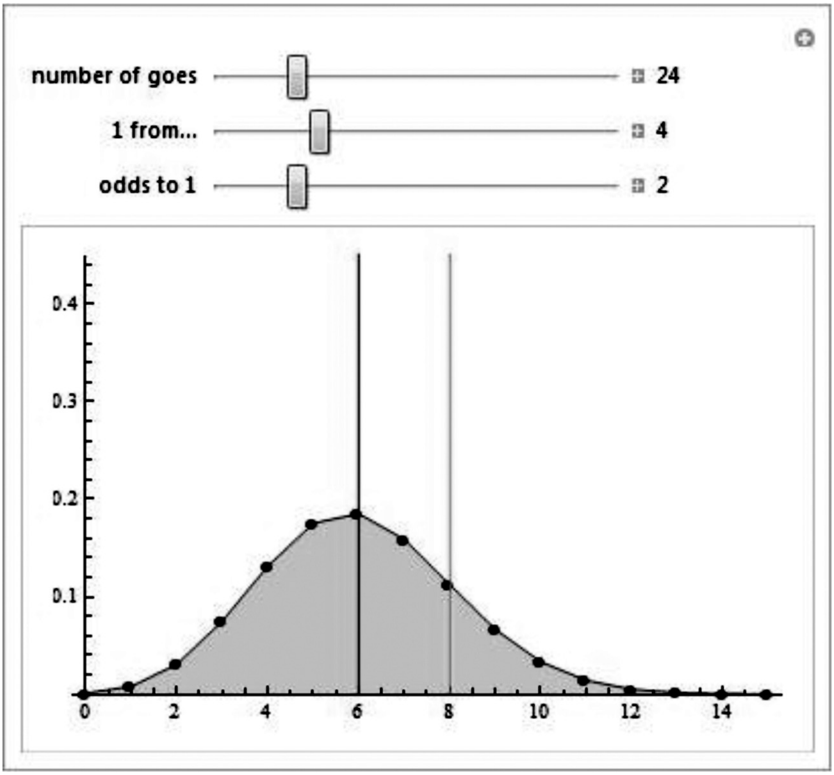

Figure 3. Expected average distribution of results from 24 goes

Secondly, programming enables us to play the game a vast number of times at the flick of a button. Actual long run outcomes may thus be found in an instant. The long run no longer takes a long time. This second remarkable dynamic display in Mathematica allows you to play the game in sets of 12, from 100 to 100,000 times, up to a total of 1.2 million games, showing the actual distribution of results of sets of 12, compared with expectation. I don't know the underlying software, but when you play 1200 games you can see variation from average expectation, but when you play 1.2 million no deviation from expectation is visible. I find it flabbergasting. It is the very point I am trying to make; how reversion to the expected loss will guarantee taking your money over time on all chance games of this sort, such as the pokies. In my view gambling provision is mostly a fraudulent activity, perpetrated on the insufficiently informed.

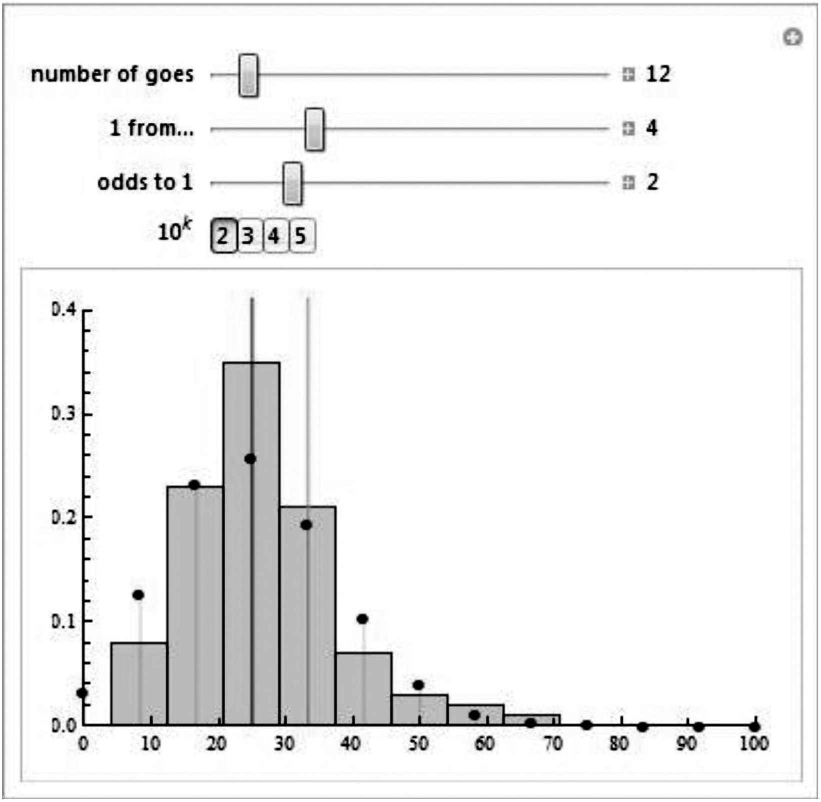

Figure 4. Expectation and distribution of wins from repeated sets of 12

More Background Mathematics for Higher and Lower Levels

A decision tree (see Figure 5) lets us see the range of possibilities from multiple goes. Although each of the 64 sequences of outcomes from 3 goes is unique, some of them share characteristics when viewed as combinations instead of permutations. Thus there are 27 ways to get 1 win from 3 goes, and so on.

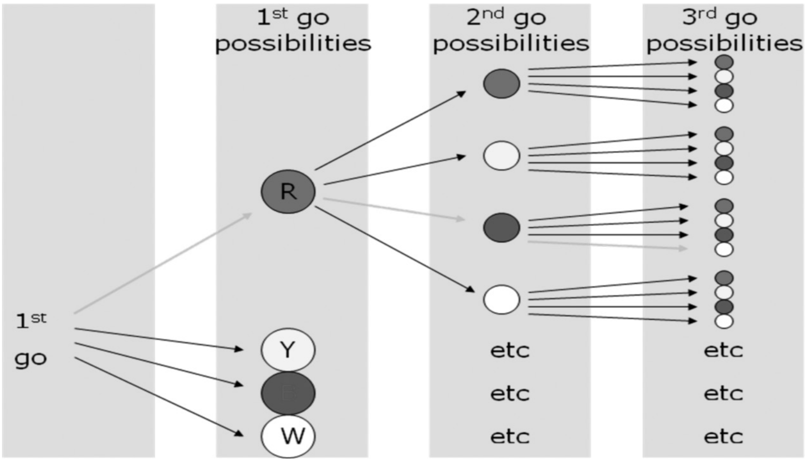

Figure 5. Mapping the Permutations of the Colours

We may also consider how the probability distribution varies as the sample size, i.e. number of games, increases.

The probability of a particular number of wins from a set number of goes is given by the formula:

$$\Pr(w) = \frac{n!}{w!(n-w)!}(p)^w\ (1-p)^{n-w}$$

Pr = probability $\quad w$ = no. of wins $\quad n$ = total no. of goes

p = pr. of a win on each go = 1 out of 4 = $\frac{1}{4}$

1 – p = pr. of a loss = 3 out of 4 = $\frac{3}{4}$ $\quad$! = factorial $\quad$ e.g.:

$$\Pr(0) = \frac{12!}{0!(12-0)!}\left(\frac{1}{4}\right)^0\ \left(\frac{3}{4}\right)^{12-0}$$

$$= \left(\frac{3}{4}\right)^{12} \quad = 0.0317$$

The most common outcome for 12 goes is 3 wins (24% of the time), giving a loss of 3 betting units. Mathematics teachers understand standard deviations, so I offer this information to impress you: After 12 goes, average loss of 3, with σ = 4.5. Quite a few winners. After 100 goes, average loss of 25, $\sigma = 13$. So, about 3% are still winners. Average loss of 100 for 400 goes, $\sigma = 26$, i.e. a winner is almost 4 standard deviations away. Not a popular place. If we played this game 400 times, a set of 400, we would need a stadium holding 10,000 players to have the likelihood that one or two players, up to a handful, might still be ahead. The likelihood of loss over increasing play is dramatic. The formula to calculate a standard deviation for a binomial distribution is given by: $\sigma = \sqrt{[np(1-p)]}$.

To compare the results from 12 goes and 36 goes, they may be graphed with matching average expectancy and breakeven lines on the two graph's axes. To make the visual comparisons easier the results are grouped in the 36 games, generating graphs with equal area under the graphs, i.e. total pr =1. The probability of getting results to the right of the breakeven line reduces as games increase, and there is increasing crowding around the average expectancy (see Figure 6).

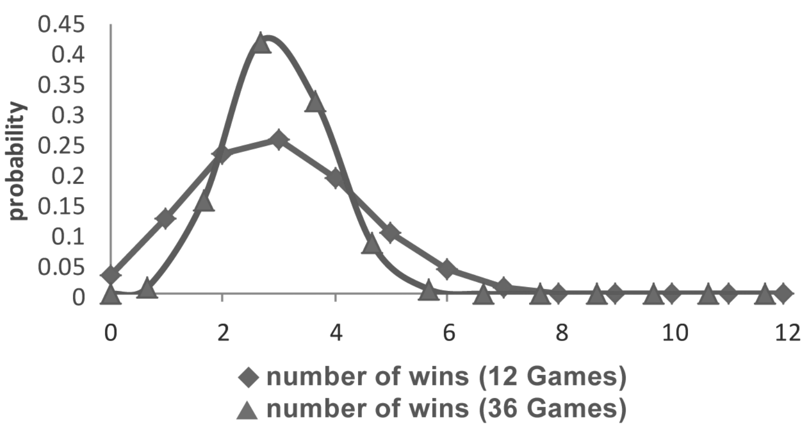

Figure 6. Probability distribution of game outcomes for 12 and 36 goes

Player results from 36 games could be graphed, with the individual movement of players between the 12 and 36 game graphs noted. Most players will regress toward the mean line.

Once the negative expectation on chance gambling is understood, it is the increasing certainty of that loss outcome over time which seems crucial to understand. Other resources can round out the teaching on distributions. Computer simulation of regression to the mean is a useful activity. Extension to gambling which is not purely chance based, such as sports betting, introduces new complications, but the difficulties in overcoming a house margin long term remains an important point.

A few other examples of teaching about the maths of gambling at a school level have impressed me. There is a Canadian website on "How Gambling Really Works", at www.

getgamblingfacts.ca. It explains the key teaching points about the maths of gambling, but is aimed at gamblers rather than school students so should be used choosily. Tim Falkiner (2001) used visual examples, such as simulation of roulette results on "even-money" (18:19) bets, in his teaching software, and some individual teachers have constructed their own long runs in Excel. Good luck!

References

Addictions *Foundation of Manitoba.* (2012). How Gambling Really Works. www.getgamblingfacts.ca

Falkiner, T. (2001). *Know the Odds.* [Computer software]. Know the Odds Inc. Melbourne.

Smith, D. (2003). *Gambling behaviour and understanding of probability concepts among university students.* In. Bragg, L. Campbell, C, G. Herbert, & J. Mousley (Eds.), Mathematics education research: Innovation, networking, opportunity. Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia. Geelong, Vic: MERGA.

Smith, D. (2011). *Graphically Explaining Expected Loss on Chance Gambling.* In L. Bragg, (Ed.), Maths is multi-dimensional. Proceedings of the Mathematical Association of Victoria 48th Annual Conference (pp. 208-218). Melbourne, Australia: MAV.

Wolfram, S. et al (2010). *Mathematica* (Version 8) [Computer software].
www.wolfram.com/mathematica

HIGH MATHEMATICAL PERFORMANCE ON CLASS TESTS IS NOT A PREDICTOR OF PROBLEM-SOLVING ABILITY: WHY?

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Drawing on my findings as a teacher and the further understandings I have developed as a researcher, I provide illustrations of confident high performing students from my research classes to explore the notion of 'enabling confidence' and 'disabling confidence'. Students with enabling confidence incline to explore unfamiliar mathematical situations and students with disabling confidence do not. The higher likelihood of many teaching practices developing disabling confidence, and our obligation to develop enabling confidence are discussed.

Introduction

Can you identify high performing students in your classes who are confident in their abilities to 'do mathematics' but need to be told: (a) what to do differently for every slight 'twist and turn' in questions in an exercise; and (b) what mathematics to use when a question is set in a different context to that in which the mathematics was learnt? Have you noticed what happens when they are asked to: (a) explore unfamiliar challenging problems; or (b) do exam questions requiring the use of familiar mathematics in unfamiliar ways? This paper examines the activity of these students in comparison to those inclined to explore mathematics.

The Nature of Confidence

My research has shown that 'resilience' in the form of Martin Seligman's (1995) 'optimism' is a characteristic of students who are inclined to explore unfamiliar mathematical ideas. It is an orientation to failures and successes which I have elaborated in terms of mathematical problem solving as 'not knowing' (failure) and 'finding out' (success). Thus, an optimistic child sees not knowing as temporary and able to be overcome through the *personal effort* of looking into the situation to identify what variables *they can control* and *which of these to vary to increase their likelihood of success—finding out more.* They perceive their *successes as permanent* (able to be achieved again) and take them on as *characteristics of self:* "I did this! I am good at this". Two of these six described facets of optimism (in italics) together characterize confidence, the "degree to which a person feels certain of her or his ability to learn and perform well in mathematics" (Hart, 1999, p. 243), and the perceived associated personal characteristic: "… one's ability to learn and to perform well on mathematical tasks" (Fenemma & Sherman, 1976, p. 326). Confident students believe they will be able to perform mathematically in the future due to personal characteristics they possess.

The Students and Their Problem Solving Responses

The four students selected, Sam, Hank, Eliza, and Patrick, illustrate the activity of students who are, or are not, inclined to explore unfamiliar mathematics. All attended the same school and were part of problem solving lessons I undertook in their classrooms during their Grade 6 year. Sam, Patrick, and Eliza were in the same class and Patrick and Eliza in the same group during the task under focus. Hank was in a different Grade 6 class in a different year. All were confident students who performed at a high level in mathematics tests in their usual mathematics classes. Although these illustrations are limited to one school and one grade, I have found such responses to tasks during my teaching (Year 7-12, see for example, Williams 1994, 1997, 2000a, 2000b, 2000c) and during my research projects (Year 12, Year 8, Grades 3-6), and can identify such activity in research undertaken by others in my classrooms (Barnes, 2000; Groves & Doig, 2004).

Sam and Hank were amongst the highest performing students in their classes on regular class tests (predominantly recall, and reproduction of taught mathematical procedures). They appeared confident in their mathematical abilities and gave their high tests marks as the reason they knew they were 'good at maths'. Each of them stated that they learnt mathematics through external means: by listening to their teacher, consulting texts, and / or finding information on the Internet.

Eliza and Patrick performed at a high level on class tests. They displayed confidence in their ability to think mathematically. They each described learning as an active process in which they participated and made sense of ideas. Patrick stated that he learnt during the problem solving lessons by working out ways to proceed where other groups had identified problems in what they had developed so far. He stated that he also learnt by thinking about why various students had made mistakes in their reports. Eliza described how concrete aides provided during the problem solving activities helped her to think as she developed an understanding of new ideas: "When I try to do things in *my mind* it is hard for me to figure it out 'till I really know how so the blocks help me to learn how to figure it out in my mind". She also described how her parents helped her when she had problems with homework: "they don't actually tell me the answer … they sort of help me on my way". Both these students saw 'not knowing' as temporary and finding out as involving developing strategies to helped them reorganize mathematical ideas as part of sense making.

The Task 'How Many Boxes'

Properties of 'boxes' as the term was used in this task (rectangular prisms/cuboids) were explored initially with students using common language to describe them. Students were then asked to find as many different solid boxes as they could that each contained 24 'little' cubes (cubic centimetres: term not provided). As they worked with this task, they were asked questions like: How many can you make? Can you see any patterns to the boxes that can be made? Why do you get that pattern? Any ideas? Have you found them all? How do you know? Can you make a mathematical argument for how you know you have them all? Once these boxes had been explored, groups were told they were going to participate in a game. They would be told how many little cubes were used to make a 'hidden box' and asked to find its dimensions (including its orientation). To do so, each group could ask one 'yes/no' question. All groups would have access to all questions and their answers, and the winning group would be the first group to give the correct dimensions of the box with an appropriate argument as to why these were the dimensions. Before the game commenced, groups had five minutes to brainstorm about what question to ask. During that time they had access to twenty-four little cubes.

Student Activity

Student task activity was captured on video with audio record of talk in each group, group reports, and group discussion of these reports. Each student took part in a post lesson video stimulated interview in which they found parts of the video they wanted to talk about, and discuss anything new they learnt and what helped them to learn it.

Sam and Hank

These two students knew the volume of a cuboid formula before the task commenced and each stated they did not learn anything new during the three-session task.

Sam, in his post-lesson interview, knew the formula included the side lengths of the box but not why multiplying them together gave the number of little cubes in the box. Sam's lack of understanding of the structure of cubes within the box was demonstrated during the task: he did not recognize there was something the matter with a 3×3×3 box presented by a member of his group. It appeared Sam did not know this box contained 27 little cubes rather than 24. As I moved around the groups, and in whole class settings, I asked him questions across more than a twenty-minute interval before Sam even began to realize there could be something the matter with the box his group member had presented. Sam's post lesson interview showed he was still unaware of the structure of a rectangular prism at the end of the task. This lack of awareness of the structure limited the ways in which he could evaluate the reasonableness of the 3 × 3 × 3 box. He would have been unable to find the number of cubes in the box by considering layers of cubes. Sam considered the task boring stating he already knew everything about this topic. This raises questions about whether Sam's confidence in his ability to know and reproduce information about volumes of cuboids limited his likelihood of recognizing task complexities that could help develop understanding of the structure that led to the rule?

Hank's activity was similar to Sam's in some regards in that he continued to work only with the rule and numerical representations through out the task. Hank quickly recognized number patterns: factors were involved. He could see that this fitted with the multiplicative nature of the volume of a rectangular prism rule. Like Sam though, although he was repeatedly asked questions about why these factors with the rule related to finding the number of cubes in the boxes, he was unable to extend his thinking beyond identifying the factors as the dimensions of the 'boxes'. One of the final questions I asked Hank was:

How does that number pattern, which you beautifully explained yesterday, fit with those actual cubes in that box- and I don't just mean length width and height. Why, when you multiply those together, do you get the total number in that box?

I then shifted away from the group and Hank repeated again what he had already said to his group (without extending his thinking further): "[fast and soft] factors are numbers that are multiple- you can multiply factors to get the number". This was a cyclical argument based around Hank's knowledge that the volume was found by multiplying the length by the width by the height, and his knowledge of the meanings of factors and multiples. Hank was thinking numerically without linking what he had identified to the internal structure of the boxes. His lack of understanding of that structure and how it related to the factors he had generated was apparent when I gave an additional clue about the hidden box during the game: "the cross section to it has nine little squares in it". Many groups were able to make sense of this clue because they had become aware of the arrangement of cubes in boxes (the structure), and the term 'cross section' had already been explained. Hank though, exclaimed: "Hu … that's weird!" as he tried to make sense of the clue linking the structure of the cubes within the box with the dimensions and the volume.

In each of these classes, by the end the task, many groups who did not know the volume formula at the start of the task had developed an understanding of the structure of a cuboid and had developed some way of finding the number of cubes within using this structure. Hank and Sam on the other hand continued to know only the volume rule and were unable to give any explanation for why it worked.

Eliza and Patrick

These two students both actively thought about relationships between the numbers, the cubes, and the boxes during the task rather than only focusing on numbers and number patterns. Eliza reported changes in her understandings over time that demonstrated her group had progressively developed new ideas, and that she had contributed to this development. When her group (including Patrick) tried to work out whether a box could be made with 32 cubes, they only had 24 little cubes available. The group started to make a cube stack (2 × 2 square cross section) counting the cubes one by one. In her interview Eliza described the strategy the group used to cope with the limited number of cubes they had available: "I can't remember how many we had stacked up- but then we had (pause) a drawing on a piece of paper [two 2 × 2 grids]". She went on to explain the purpose of the drawing: "we needed that to pretend there was another bit of eight [two layers of four to add to the stack]". At that stage, the group did not know how many layers they needed. It was Eliza's idea to represent the rest of the cubes on paper. In her interview, Eliza identified that it was during this time of making extra layers on paper that the group realized they could count by fours (layers) to find the number of cubes in the box. Eliza displayed her new understandings in her report to the class in which she paused / hesitated at times over terms she and her group had recently begun to use. By the time she reported, she had realized she could change the orientation of the box to four layers of eight cubes:

> Start by making four flat boxes out of eight one-centimetre cubes. Stack the four to make 32. Count them- there should be four in the height and eight in the length so you use four times eight, which is thir- so you have 32 in the box

Eliza gave the length as eight but her explanations to a student questioning this use of the term 'length' showed she meant the base contained eight little cubes—the group were still in the process of developing appropriate terminology. Eliza contributed several times to her group's construction of new knowledge. Through this process she and other group members developed an understanding of the structure of the cubes in the 'box' to the extent of 'seeing' layers but not yet 'seeing' the base as an array (see Williams, 2010 for more about problems students have 'seeing' these arrays).

Eliza's responses show that trying to work out what is unfamiliar ('not known') is something she does frequently, and that she expects to be able to gain some success in doing so. She perceived that the personal effort of using her mind while using the blocks would help her. When asked how she thought she was 'going in maths' and how she decided that, Eliza reflected for some time before saying: "I know it's not because (pause) I get things right" and after further consideration she continued: "I think it's because I contribute to the question and things (pause) rather than (pause) just agree and disagree".

Patrick explained that he learnt by thinking about other group reports including: how to do something another group was still trying to work out, and thinking about why another group had developed an answer that he could see was incorrect. He was willing to look in to situations to see what could be changed to increase the likelihood of finding out about something that 'was not yet known'. For example, he used his newly developed understanding of layers to find a way to solve what another group was puzzling about. That group had intended to make a box containing 12 cubes and found the box they produced contained 24 cubes. In his interview, Patrick reported his thinking on how they could solve their problem: "You know how they got it wrong- it made me think about (pause) how they could get it right":

> It [the shape they made] was 2 2 6 [dimensions two by two by six]. They got 24 and they have to get 12 what if they changed the 6 to 3 and that would just halve it and instead of 24 you get 12.

Patrick considered the shape as six layers of four and halving the number of layers giving half the number of little cubes. Like Eliza, Patrick enacted optimistic problem solving activity.

Discussion and Conclusions

All four students displayed the pair of optimistic indicators associated with confidence, but when considering other features of optimism, Sam and Hank perceived 'finding out' as requiring the assistance of external sources (teacher, books, internet), which is a non-optimistic indicator. Each considered they knew the topic well and did not need to know more about it. Eliza and Patrick, on the other hand, perceived not knowing as temporary and were inclined to expend personal effort in looking in to situations because they knew they would be able to find out more. These are optimistic indicators. The activity of these four students shows some confident high performing students (Sam, Hank) judge their ability to do mathematics through external means like the marks they achieve, and others (Patrick, Eliza) judge their ability to do mathematics internally, by the actions they take to find out what they still do not know. Where Sam and Hank considered learning mathematics as 'taking in' rules and procedures from external sources, memorizing them and substituting into them without thinking about their meanings, Patrick and Eliza considered learning mathematics was primarily about puzzling over unfamiliar mathematical ideas and making meaning from them. Eliza and Patrick developed their confidence through the success they gained from working things out for themselves (enabling confidence), and Hank and Sam developed their confidence from their high marks on tests that were primarily about reproducing mathematics they had been taught (disabling confidence). These illustrations show what I have found more generally across upper elementary and secondary school classes: the way students develop confidence can influence their inclination to problem solve.

Research has linked confidence with ability to do mathematics, but ability to do mathematics was measured by performance on multiple choice items not open-ended tasks (Pajares & Miller, 1997). This raises several questions: What is the nature of the ability to do mathematics? Do we need to look more deeply at the nature of confidence and mathematical performance to be able to interpret associations between them?

Given many mathematics classes across the state (and internationally) still use teaching approaches that focus primarily on memorizing rules, reproducing rules, and using procedures to answer questions that students have already been taught how to solve, there is a high likelihood that many students in our classes are developing disabling confidence. Where assessment *only* values such mathematical performances, the problem is amplified. If we want to build the problem solving capacity of students, we need to give them opportunities to think for themselves and reward them for doing so. In this way, the ability to think mathematically rather than just repeat rules and procedures becomes the intended goal and our students become better problem solvers who understand where it is appropriate to use what mathematics. This is our challenge! The shaping paper for the Australian Mathematics Curriculum highlights the need for students to develop deep mathematical understandings (see for example, Williams, 2010). We must not lose sight of this intention as we plan the future of mathematics education in Victoria.

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References

Barnes, M. (2000). *'Magical' moments in mathematics: Insights into the process of coming to know.* For the Learning of Mathematics, 20(1), 33-43.

Groves, S., & Doig, B. (2004). *Progressive discourse in mathematics classes—the task of the teacher.* In M. Hoines & A. Berit Fuglestad (Eds.), Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education (pp. 495-502). Bergen, Norway: PME.

Hart, L. (1989). *Classroom processes, sex of student, and confidence in learning mathematics.* Journal for Research in Mathematics Education, 20(3), 242-260.

Pajares, F., & Miller, D. (1997). *Mathematics self-efficacy and mathematical problem solving: Implications of using different forms of assessment.* The Journal of Experimental Education, 65(3), 213-228.

Seligman, M. (with Reivich, K., Jaycox, L., Gillham, J.). (1995). *The optimistic child.* Adelaide: Griffin Press.

Williams, G. (2010). *Australian Mathematics Curriculum implementation: Teacher reflections on student thinking informing future directions.* In M. Westbrook, D. Treeby, C. Sexton et al. (Eds.), –New- Curriculum –New- Opportunities, Proceedings of the 2010 MAV Conference, (pp. 187-198). Brunswick, Victoria: Mathematical Association of Victoria.

Williams, G. (2000a). *Angles in parallel lines: Engaged and learning about learning.* In J. Wakefield (Ed.), Mathematics: Shaping the future (pp. 395-407). Brunswick, Victoria, Mathematical Association of Victoria.

Williams, G. (2000b). *Investigating volumes of cuboids.* In J. Wakefield (Ed.), Mathematics: Shaping the future (pp. 136-149). Mathematical Association of Victoria.

Williams, G., Peck, R., Cowling, G., & Ripon, J. (2000c). *Mathematical Methods 1/2/3/4:*

tasks designed to develop the characteristics possessed by talented mathematicians. In J. Wakefield (Ed.), Mathematics: Shaping the future (pp. 408-421). Brunswick, Victoria, Mathematical Association of Victoria.

Williams, G. (1997). *Creating an atmosphere conducive to learning: A small group/class feedback model.* In N. Scott and H. Hollingsworth (Eds.), Mathematics: Creating the future (pp. 354-361). Victoria, AAMT.

Williams, G. (1994). *The double derivative: Students learning rather than teacher teaching.* C. Beesey and D. Rasmussen (Eds), Mathematics without limits (pp. 447-453). Brunswick, Victoria: Mathematical Association of Victoria.

PROBLEM-SOLVING: 'SAME PACE OF THINKING' GROUPS

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Gaye and Brunswick South West Primary School staff have worked together since 2004 on problem solving in mathematics. We have explored grouping students, questioning, and selecting adapting and designing tasks. In this paper, Sharon, Judy (Grade 5/6 teachers), and Gaye focus on the benefits of composing groups that enable rich student conversations during the development of new mathematical ideas ('same pace of thinking' groups) and strategies to do so.

Introduction

Terezinha Nunes (2010) studied young Brazilian street vendors' mathematical competencies at school and on the street as they managed and sold their wares. Selling on the streets, they competently worked out profits, and made decisions about what they bought and sold but at school they were unable to perform set calculations based on the same mathematics. 'School mathematics', which was presented as rules and procedures, was meaningless to them. They were failing at school yet they understood this same mathematics in out of school contexts where they developed mathematical ideas for purposes that were extremely important for their livelihood. The Australian Mathematics Curriculum as framed was intended to address the problem Nunes identified. That students would learn mathematics in meaningful ways that include problem-solving and student conversations.

Gaye and Brunswick South West Primary School (BSW) staff have focused on learning mathematics this way for several years now. Williams and Cavellin (2004) describe early work, and Judy, Sharon and Gaye now reflect on more recent work together in Grade 5/6 classrooms. Their focus is on group composition: benefits of grouping students by 'same pace of thinking', and strategies to enable this.

Engaged to Learn Approach

'Same pace of thinking'groups is one of the key aspects of Gaye's Engaged to Learn Approach to problem solving (Williams, 2009) within which students work in small groups (3-4 students) with problem solving tasks that are accessible through simple or more complex mathematics that students can develop using a variety of representations. Each group reports to the class at regular intervals with a different reporter each time. The teacher does not give hints, agree or disagree with mathematics developed but rather asks questions to elicit further student thinking. The teacher composes the groups using criteria Gaye developed as a teacher:

- Preferably groups of four but three if this is not possible
- Gender balance if possible, but at least no less girls than boys
- In general separate friends so previously developed social interactions do not interfere with collaborations around creative thinking
- 'Positive personality' to overcome any negative influence
- Same pace of thinking (not same level of performance)

After observing her students working with Gaye, Judy shared the following insights on how she perceived the Engaged to Learn Approach structured student learning. She could see that this classroom activity increases the likelihood of equal participation including equal 'voice' because:

- Each individual in the group is aware of his/her roles, and knows that they will be reporting to the class during the session.
- This eliminates the option to sit back and let others do the work.
- Students pay attention throughout to be able to report confidently.
- The group 'priming' the reporter reduces pressure on the reporter.
- Less confident students don't tend to feel threatened during their reports because of this group preparation (priming).

'Priming' involves the groups brainstorming what to report, the reporter practicing this report, and the group refining it to fit what they want. Judy links this reporting process to Joseph Joubert: "to teach is to learn twice" <http://thinkexist.com/quotation/to_

teach_is_to_learn_twice/7984.html> because all group members have the opportunity to reinforce concepts as they prepare and report. She could see that students had developed and consolidated knowledge during reporting rounds and that it assisted them to move forward.

Judy identified instructions, materials, and timing, as key to the approach. She noted that clear instructions were given about the nature of the task at the start of sessions but information about *how* to approach the task is not given. Various materials were made available to students: some possibly useful and some not. Students decide how to approach the problem, what to use, and how to use it, so the usefulness (or not) of materials available could depend on the solution pathways selected. By not telling students what materials to use or how to use them, the problem is left very open-ended with many opportunities for groups to pursue pathways they consider useful. Judy emphasized the need for the teacher to carefully consider what materials to present, and how. She identified several strategies she had seen Gaye use and why she considered them useful. By giving students insufficient pieces of concrete materials for each to create an individual model, students were 'forced' to collaborate, and by providing only one piece of paper per group, they were encouraged to work together in the middle of the table. Judy also drew attention to the clear time limits (and warnings as limits approached) contributing to sustained focus. Because time intervals were short, and students were keen to have something to show in their reports, they focused. They didn't want to stand up and say nothing. During reporting time, students were ready to share, and to listen to what others had developed and consider how it fitted with their own ideas.

Judy provided Gaye with new insight into how this approach satisfied (whilst modifying) the behavior of a good thinker who wanted to dominate normal classroom discussions. Within this approach, he had his chance to speak in the group, but also had to listen to others, especially when he was reporting and required to represent the whole group. Because others in his group thought at a similar pace, they also wanted to be heard not sit back and say nothing, so he could not dominate. Judy noticed this very competitive student listened carefully to other group presentations to gain "clues" to a puzzle that he was determined to solve to be the "winner". In looking at her research videos from various schools, Gaye found other examples of such students. Thanks Judy!

Sharon drew attention to the careful wording of tasks and reflected that the teacher needs to have a clear idea of the *desired* learning as part of ensuring that maximum learning is achieved. She emphasised the process as just as important as the product: "providing students with opportunities to explore, and share their ideas has been much more valuable than racing to an immediate 'correct' answer". Sharon illustrates with what she observed as her own class worked with Gaye. The task required students to make solid boxes using 24 tiny cubes. Students had time to experiment then describe one of their self-designed boxes to the class who were unable to see it clearly due to its size. Sharon reflected that, instead of rushing straight to an answer the students were:

- Encouraged to experiment with different possibilities/models, and
- Given freedom to develop their own mathematical language without the pressure of instantly responding with 'correct' terminology.

Her perception of the role of a teacher changed significantly whilst working in her classroom with Gaye. She drew attention to the following comment Gaye made to her in class as a turning point in changing her mind-set: "there is more in what is not said [by the teacher]". Sharon began to reflect on this comment and its implications: moving away from teacher in control shepherding students to the answer, and towards the teacher stepping back and taking a more passive role. Sharon elaborated further: "instead of hinting the teacher simply listens and encourages the students to ask questions and reflect on their own and others' thinking. Instead of leading the students to a solution the teacher asks questions that prompt discussion and deepen thinking." Sharon emphasised that "You cannot assume that students will come to a task with a definite set of skills or prior knowledge". Instead, different students come to the task with different knowledge. Gaye draws attention to Sharon's initial impression of a 'more 'passive' teacher which is a common first impression for many teachers (e.g., see Williams, Menzel & Sheridan, 2009). She emphasises the complexity of teacher actions involved in listening to the students, then flexibly responding to their moment-by-moment activity with highly appropriate questions: responsive action not passive inaction.

Group Composition

Sharon has changed some of her grouping practices whilst participating in the Engaged to Learn Approach. She composes smaller groups, pays attention to gender balance, sometimes reconfigures seating arrangements, and has developed strategies for grouping in classes with more boys than girls. She has realised that larger groups do not work as successfully as groups of three or four because some students tend to sit back and not participate as actively whereas groups of three or four give all students more chance to participate. She also observed that students are less intimidated when sharing ideas in smaller groups. Sharon found that something as simple as seating allocation within the group has effects. For example, seating students of the same gender on one side of the table can help a student to feel more comfortable, but allowing friends to sit side by side or directly across from each other can contribute to excluding other students from discussions. Sitting friends diagonally opposite each other can help resolve this. Seat allocation can also be used to limit the talk of dominant students and open opportunities for quieter students to talk. Sharon observed effects of non-gender balance when one girl was absent leaving one girl with two boys. This girl participated far less in that session than in those where the other girl was present. In 2012 when Grade 5/6 had more boys than girls, Sharon carefully considered the overall confidence of different girls to decide who could comfortably work in a 'mostly boy' group. She also created several groups of three to accommodate for the gender imbalance. Useful strategies! Gaye adds another: some 'all boy' groups.

To limit disengagement, Sharon sometimes places negative, less enthusiastic students with more positive students. She recalls grouping a fast paced thinker, who tended to be negative in new situations, with resilient, highly enthusiastic students. Sharon describes 'resilient' as taking everything in their stride - not phased by a student's repeated attempts to distract. This group ignored this student's negative comments and encouraged him to share his ideas and become involved in the task. They focused on what he was able to share instead of commenting on his negative activity.

Gaye describes groups with the same pace of thinking as able to 'think together' during problem solving activity: understanding and building on ideas as others suggest them and contributing ideas themselves. A student from Gaye's research captures how many students perceive 'same pace of thinking' groups:

> [The task] seemed a little bit … daunting at first … but once we started I thought oh well this isn't so hard- … in some maths classes … we have to learn one thing … everyone has to learn the same thing … but … with this lesson … I liked it because … we could kind of choose what we wanted to get from it- … yeah … I like the freedom … definitely … because we got to choose what we … get out of it-

This student identified that each group could focus on mathematics that was useful for them. She liked learning this way with freedom to select what was valuable for the group to pursue rather having to follow the teacher's lead.

Judy has found that in 'same pace of thinking' groups, students can't rely on one person to do all the work or make suggestions. They need to build ideas together to produce a report. She groups quieter students together because she knows the expectations of the approach mean each quiet student must contribute. They cannot sit back and rely on another because there is no one in the group who would normally do the talking and they know they have to report their group findings. This need to produce findings, and the absence of vocal group members motivates these students to speak. Judy found it rewarding to observe many moments where these students had the opportunity to speak up and be heard (probably for the first time), had something valuable to say, *and* saw it acknowledged by others. In forming her 'same pace of thinking' groups, she considers the qualities of students such as how vocal they are, how confident, how easily they are distracted, and how deeply they think. She is inclined to put very vocal students together so that one student can't just do all the talking but instead group members compete and debate with each other regardless of their usual performance in maths. Judy also discusses those students with a 'good grasp of maths' who tend to sit back in 'normal' group work in class and say nothing. She suggests possible reasons: they see no need as the answer appears obvious, or the teacher won't choose them because they always know the answer, or they are just disengaged because they don't feel challenged. During the Engaged to Learn Approach, Judy found these students participated regardless of their prior achievements and knowledge (as she understood that knowledge from what she generally saw in class). She thought this was because students were challenged to find out what they could about the given situation, and they didn't feel pressured about what they "know" or "should" know. They were confident to start or enter the 'conversation' at any point. Notice how Judy qualifies what she knows of students' knowledge in terms of what she has seen in 'usual' maths classes. Like many teachers working with Gaye, there were times when Sharon and Judy were really surprised about the thinking some students did.

Sharon observed that students had more control of their learning in 'same pace of thinking' groups (like the student Gaye quoted). She found these groups allow students to develop a common understanding and to reach a solution collaboratively as they share and explore ideas at their own pace and choose appropriate pathways that interest them. They have the choice of different materials and the *class* 'guide' each other's thinking through the staggered reporting throughout the sessions. She was particularly interested in what occurred in 'slower pace of thinking' groups and how to support them. Sharon considered herself lucky to have opportunity to observe and reflect on one of her slower pace of thinking groups achieving success working as a team. These students who often sat back, afraid of being wrong or unsure where to start when faced with unfamiliar mathematical ideas, supported each other taking risks, experimenting and explaining their thinking. They openly discussed their ideas and enthusiastically trialed each other's. They were surprisingly keen to present their findings to the class and listened for clues from other reports to assist themselves. Sharon identified the teacher's challenge as 'not to guide'in a certain direction but to ask questions to prompt further discussion and encourage further exploring of their ideas. She found that locating herself to the side of a group simply listening had a great impact. Students felt supported but with the freedom to work independently.

When Sharon and Gaye decided to group together students likely to 'take over' in other groups (Judy's vocal students), Sharon tried to dissect why one of these groups worked well together when we had expected they might be 'dysfunctional'. Gaye draws attention to sometimes needing to compose one 'dysfunctional group' to achieve other 'good' groups, and the need to ensure these students are in 'good groups' for the next task. With the group in question, Sharon considered these students worked comfortably together because they shared the same abstract thinking. They harnessed their combined energy and became highly enthusiastic as they experimented with each other's ideas, welcomed new suggestions, and actively engaged in animated debate about possible effective strategies.

With regard to forming 'same pace of thinking' groups, Sharon reflects that even after observing and participating in classes with Gaye, there is always a temptation to revert back to familiar performance based groupings where you simply look at *data*. Gaye can see why busy teachers sometimes feel this temptation. All that said, Sharon has developed very useful strategies to help form 'same pace of thinking' groups in 2012. In 2011, she worked with Gaye with a BSW Grade 5/6 class and in 2012 many of her Grade 6 students were Grade 5 students in that 2011 class. Sharon uses groupings she made with Gaye in 2011 to form her 2012 groups. She then looks closely at her new Grade 5 students, thinks carefully about their pace of thinking whilst trying to equate each to similar 2011 Grade 6 students (now in Year 7). She finds this enables her to step into Gaye's mindset and think carefully about her students' paces of thinking without worrying about getting it all terribly wrong. This has helped her think about the dynamics of successful groups and reflect on groups that had not worked as well as expected in 2011. Gaye says that by using these strategies, Sharon is putting together a 'bank of ideas' to use in the future. Even with Gaye's groupings from last year to rely on, Sharon found it was a case of trial and error at first and *refinement after she observed the students working together.* She says that even after working so closely with Gaye, she still had a tendency to question her own decisions but she perseveres. Sharon pays particular attention to changes in student's personalities over time, and differences in individual's paces of thinking. She regularly evaluates her groups looking for ways to improve them. Gaye says what Sharon shows is a willingness to experiment, trial ideas, reflect, and modify and that such activity will enable Sharon to more confidently form same pace of thinking groups over time.

Sharon was initially amazed at how different the composition of Gaye's same 'pace of thinking groups' were from her usual performance-based groups. She reported that BSW upper primary staff now use a mix of explicit, performance-based groupings and 'same pace of thinking' groupings:

> "We believe this allows the best of both worlds for our students. Through the use of fluid groupings we deliver explicit instruction based on the student's data which targets their Zone of Proximal Development. Through the use of 'same pace of thinking' group we provide students with the opportunity to work with students from a range of performance levels on more open-ended tasks."

These BSW teachers feel this gives groups opportunities to choose their own pathways to achieve a common learning goal in more personalized setting.

Conclusions

The process of learning to place students in 'same pace of thinking' groups involves attention to personalities, student conversations, and composing and arranging groups to limit the talk of dominant students and amplify the talk of quieter students. Most importantly, forming these groups is a process of trying what looks likely to work, identifying what worked well and what needs to change, and trying again. Judy and Sharon's strategies give you a 'head start' to such trial and error. We hope you are surprised and delighted with the thinking you 'hear'.

References

Nunes, T. (2010). *Learning outside of school.* In P. Peterson, E. Baker, & B. McGaw (Eds.), International encyclopedia of education (3rd ed.) (pp. 457–463). Amsterdam, The Netherlands: Elsevier.

Williams, G. (2009). *A deep learning approach.* Shine, 9, 52-53.

Williams, G., Menzel, B., & Sheridan, B. (2009). Suddenly we had engaged Middle Years students! How did that happen? Mathematics of Prime Importance, Proceedings of the 2009 MAV Conference, pp. 279-287. Mathematical Association of Victoria, Melbourne.

Williams, G., & Cavellin, N. (2004). *Teacher researcher can enrich teaching practice: An example.* In B. Tadich, S. Tobias, C. Brew, B. Beatty & P. Sullivan. Towards excellence in mathematics (pp. 587-597). Brunswick, Victoria; Mathematical Association of Victoria.

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